

On algebraic semigroups and monoids

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Abstract

We present some fundamental results on (possibly nonlinear) algebraic semigroups and monoids. These include a version of Chevalley's structure theorem for algebraic groups in the setting of irreducible algebraic monoids, and the description of algebraic semigroup structures on curves and complete varieties.

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1 Introduction

Algebraic semigroups are defined in very simple terms: they are algebraic varieties endowed with a composition law which is associative and a morphism of varieties. So far, their study has focused on the class of linear algebraic semigroups, that is, of closed subvarieties of the space of $n \times n$ matrices that are stable under matrix multiplication; note

that for an algebraic semigroup, being linear is equivalent to being affine. The theory has been especially developed by Putcha and Renner for linear algebraic monoids, i.e., those having a neutral element (see the books [Pu88, Re05]).

In addition, there has been recent progress on the structure of (possibly nonlinear) algebraic monoids: by work of Rittatore, the invertible elements of any irreducible algebraic monoid M form an algebraic group $G(M)$, open in M (see [Ri98, Thm. 1]). Moreover, M is linear if and only if so is $G(M)$ (see [Ri07, Thm. 5]). Also, the structure of normal irreducible algebraic monoids reduces to the linear case, as shown by Rittatore and the author: any such monoid is a homogeneous fiber bundle over an abelian variety, with fiber a normal irreducible linear algebraic monoid (see [BR07, Thm. 4.1], and [RR11] for further developments). This was extended by the author to all irreducible monoids in characteristic 0 (see [Br08, Thm. 3.2.1]).

In this article, we obtain some fundamental results on algebraic semigroups and monoids, that include the above structure theorems in slightly more general versions. We also describe all algebraic semigroup structures on abelian varieties, irreducible curves and complete irreducible varieties. The latter result is motivated by a remarkable theorem of Mumford: if a complete irreducible variety X has a (possibly nonassociative) composition law μ with a neutral element, then X is an abelian variety with group law μ (see [Mu74, Chap. II, §4, Appendix]).

As in [Pu88, Re05], we work over an algebraically closed field of arbitrary characteristic. But we have to resort to somewhat more advanced methods of algebraic geometry, as the varieties under consideration are not necessarily affine. For example, to show that every algebraic semigroup has an idempotent, we use an argument of reduction to a finite field, while the corresponding statement for affine algebraic semigroups follows from linear algebra. Also, we occasionally use some semigroup and monoid schemes, but did not endeavour to study them systematically.

This text is organized as follows. Section 2 presents general results on idempotents of algebraic semigroups and invertible elements of algebraic monoids. Both topics are fairly interwoven: for example, the fact that every algebraic monoid having no nontrivial idempotent is a group (whose proof is again more involved than in the linear case) implies a version of the Rees structure theorem for simple algebraic semigroups. In Section 3, we show that the Albanese morphism of an irreducible algebraic monoid is a homogeneous fibration with fiber an affine monoid scheme. This generalization of the main result of [BR07] is obtained via a new approach, based on the universal homomorphism from the monoid to an algebraic group. In Section 4, we describe all semigroup structures on certain classes of varieties. We begin with the easy case of abelian varieties; as an unexpected consequence, we show that all the maximal submonoids of a given irreducible algebraic semigroup have the same Albanese variety. Then we show that every irreducible semigroup of dimension 1 is either an algebraic group or an affine monomial curve; this generalizes a result of Putcha in the affine case (see [Pu80, Thm. 2.13] and [Pu81, Thm. 2.9]). Next, we describe all complete irreducible semigroups, via another variant of the Rees structure theorem. Finally, we show (in loose words) that the automorphisms of a complete variety are open and closed in the endomorphisms. This rigidity result has applications to complete algebraic semigroups, and yields another approach to

the above theorem of Mumford.

This article makes only the first steps in the study of (possibly nonlinear) algebraic semigroups and monoids, which yields many open questions. From the viewpoint of algebraic geometry, it is an attractive problem to describe all algebraic semigroup structures on a given variety. Our classes of examples suggest that the associativity condition may impose strong restrictions which make this problem tractable: for instance, the composition laws on the affine line are of course all the polynomial functions in two variables, but those that are associative turn out to be isomorphic to the maps $(x, y) \mapsto 0$, x , y or xy . From the viewpoint of semigroup theory, it is natural to investigate the structure of an algebraic semigroup in terms of its idempotents and the associated (algebraic) subgroups. Here a very recent result of Renner and the author (see [BR12]) asserting that every algebraic semigroup S is strongly π -regular (i.e., for any $x \in S$, some power x^m belongs to a subgroup) opens the door to further developments.

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Notation and conventions. Throughout this article, we fix an algebraically closed field k . A *variety* is a reduced, separated scheme of finite type over k ; in particular, varieties need not be irreducible. By a *point* of a variety X , we mean a closed (or equivalently, k -rational) point; we may identify X to its set of points equipped with the Zariski topology and with the structure sheaf \mathcal{O}_X . *Morphisms* of varieties are understood to be k -morphisms.

The textbook [Ha77] will be our standard reference for algebraic geometry, and [Ei95] for commutative algebra. We will also use the books [Sp98] and [Mu74] for some basic results on linear algebraic groups, resp. abelian varieties.

2 Algebraic semigroups and monoids

2.1 Basic definitions and examples

Definition 2.1.1. An (abstract) *semigroup* is a set S equipped with an associative composition law $\mu : S \times S \rightarrow S$. When S is a variety and μ is a morphism, we say that (S, μ) is an *algebraic semigroup*.

A *neutral (resp. zero) element* of a semigroup (S, μ) is an element $x_o \in S$ such that $\mu(x, x_o) = \mu(x_o, x) = x$ for all $x \in S$ (resp. $\mu(x, x_o) = \mu(x_o, x) = x_o$ for all $x \in S$).

An abstract (resp. algebraic) semigroup (S, μ) equipped with a neutral element x_o is called an abstract (resp. algebraic) *monoid*.

An *algebraic group* is a group G equipped with the structure of a variety, such that the group law μ and the inverse map $\iota : G \rightarrow G$, $g \mapsto g^{-1}$ are morphisms.

Clearly, a neutral element x_o of a semigroup S is unique if it exists; we then denote x_o by 1_S , or just by 1 if this yields no confusion. Likewise, a zero element is unique if it

exists, and we then denote it by 0_S or 0 . Also, we simply denote the semigroup law μ by $(x, y) \mapsto xy$.

Definition 2.1.2. A *left ideal* of a semigroup (S, μ) is a subset I of S such that $xy \in I$ for any $x \in S$ and $y \in I$. *Right ideals* are defined similarly; a *two-sided ideal* is of course a left and right ideal.

Definition 2.1.3. Given two semigroups S and S' , a *homomorphism of semigroups* is a map $\varphi : S \rightarrow S'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. When S and S' are monoids, we say that φ is a *homomorphism of monoids* if in addition $\varphi(1_S) = 1_{S'}$.

A *homomorphism of algebraic semigroups* is a homomorphism of semigroups which is also a morphism of varieties. Homomorphism of algebraic monoids, resp. of algebraic groups, are defined similarly.

Definition 2.1.4. An *idempotent* of a semigroup S is an element $e \in S$ such that $e^2 = e$. We denote by $E(S)$ the set of idempotents.

Idempotents yield much insight in the structure of semigroups; this is illustrated by the following:

Remarks 2.1.5. (i) Let $\varphi : S \rightarrow S'$ be a homomorphism of semigroups. Then φ sends $E(S)$ to $E(S')$; moreover, the fiber of φ at an arbitrary point $x' \in S'$ is a subsemigroup of S if and only if $x' \in E(S')$.

(ii) Let S be a semigroup, and $M \subseteq S$ a submonoid with neutral element e . Then M is contained in the subset

$$\{x \in S \mid ex = xe = x\} = \{exe \mid x \in S\} =: eSe,$$

which is the largest submonoid of S with neutral element e . This defines a bijective correspondence between idempotents and maximal submonoids of S .

(iii) Let S be a semigroup, and $e \in E(S)$. Then the subset

$$Se := \{xe \mid x \in S\} = \{x \in S \mid xe = x\}$$

is a left ideal of S , and the map

$$\varphi : S \longrightarrow Se, \quad x \longmapsto xe$$

is a *retraction* (i.e., $\varphi(x) = x$ for all $x \in Se$). The fiber of φ at e ,

$$S_e := \{x \in S \mid xe = e\},$$

is a subsemigroup of S . Moreover, the restriction

$$\psi := \varphi|_{eS} : eS \longrightarrow eS \cap Se = eSe$$

is a *retraction of semigroups*, that is, $\varphi(x) = x$ for all $x \in eSe$ and φ is a homomorphism (indeed, $xeye = xye$ for all $x \in S$ and $y \in eS$). The fiber of φ at e ,

$$eS_e := \{x \in S \mid ex = x \text{ and } xe = e\},$$

is a subsemigroup of S with law $(x, y) \mapsto y$ (since $xy = xey = ey = y$ for all $x, y \in eS_e$).

When S is an algebraic semigroup, $E(S)$ is a closed subvariety. Moreover, Se , S_e , eSe and eS_e are closed in S as well, and φ (resp. ψ) is a retraction of varieties (resp. of algebraic semigroups). In particular, every maximal abstract submonoid of S is closed.

Similar assertions hold for the right ideal eS and the subsemigroups

$${}_eS := \{x \in S \mid ex = e\}, \quad {}_eSe := \{x \in S \mid xe = x \text{ and } ex = e\}.$$

An abstract semigroup may have no idempotent; for example, the set of positive integers equipped with the addition. Yet we have:

Proposition 2.1.6. *Any algebraic semigroup has an idempotent.*

Proof. We use a classical argument of reduction to a finite field. Let (S, μ) be an algebraic semigroup, and choose $x \in S$. Then S , μ and x are defined over some finitely generated subring R of the ground field k .

If k is the algebraic closure of a prime field \mathbb{F}_p , then R is a finite field \mathbb{F}_q for some power q of p . Thus, the powers x^n , where n runs over the positive integers, form a finite subsemigroup of S . We claim that some x^n is idempotent. Indeed, we have $x^a = x^b$ for some integers $a > b > 0$. Thus, $x^b = x^{b+m(a-b)}$ for all $m > 0$. In particular, $x^b = x^{b(a-b+1)}$. It follows that $x^{b(a-b)}$ is idempotent.

For an arbitrary field k , we have a semigroup scheme $f : S_R \rightarrow \text{Spec}(R)$, its closed subscheme $g : E(S_R) \rightarrow \text{Spec}(R)$ of idempotents, and a section x of f . Let \mathfrak{m} be a maximal ideal of R ; then R/\mathfrak{m} is a finite field. By the previous step, $E(S_R)$ has points over R/\mathfrak{m} . So the image of g contains all closed points of $\text{Spec}(R)$. But since R is noetherian and the morphism g is of finite type, the image of this morphism is constructible (see e.g. [Ha77, Exer. II.3.19]). Thus, this image contains the generic point of $\text{Spec}(R)$ (see e.g. [loc. cit., Exer. II.3.18]), i.e., $E(S_R)$ has points over the fraction field of R . Hence $E(S)$ has (closed) points over the larger algebraically closed field k . \square

Combining the above proposition with Remark 2.1.5 (i), we obtain:

Corollary 2.1.7. *Let $f : S \rightarrow S'$ be a surjective homomorphism of algebraic semigroups. Then $f(E(S)) = E(S')$.*

We now present several classes of (algebraic) semigroups:

Examples 2.1.8. (i) Any set X has two semigroup laws μ_l , μ_r given by $\mu_l(x, y) := x$ (resp. $\mu_r(x, y) := y$) for all $x, y \in X$. For both laws, every element is idempotent, and X has no proper two-sided ideal.

Also, every point $x \in X$ defines a semigroup law μ_x by $\mu_x(y, z) := x$ for all $y, z \in X$. Then x is the zero element; it is the unique idempotent, and the unique proper two-sided ideal as well.

The maps μ_l , μ_r , μ_x ($x \in X$) will be called the *trivial semigroup laws* on X . When X is a variety, these maps are algebraic semigroup laws. Note that every morphism of varieties $f : X \rightarrow Y$ is a homomorphism of algebraic semigroups $(X, \mu_r) \rightarrow (Y, \mu_r)$, and likewise for μ_l . Also, f is a homomorphism $(X, \mu_x) \rightarrow (Y, \mu_y)$, where $y := f(x)$.

(ii) Let X be a set, $Y \subseteq X$ a subset, $\rho : X \rightarrow Y$ a retraction, and ν a semigroup law on Y . Then the map

$$\mu : X \times X \longrightarrow X, \quad (x_1, x_2) \longmapsto \nu(\rho(x_1), \rho(x_2))$$

is easily seen to be a semigroup law on X . Moreover, ρ is a retraction of semigroups, and $E(X) = E(Y)$. If in addition X is a variety, Y is a closed subvariety and ρ, ν are morphisms, then (X, μ) is an algebraic semigroup.

When Y consists of a single point x , we recover the semigroup law μ_x of the preceding example.

(iii) Given two semigroups (S, μ) and (S', μ') , we may define a composition law ν on the disjoint union $S \sqcup S'$ by

$$\nu(x, y) = \begin{cases} \mu(x, y) & \text{if } x, y \in S, \\ y & \text{if } x \in S \text{ and } y \in S', \\ x & \text{if } x \in S' \text{ and } y \in S, \\ \mu'(x, y) & \text{if } x, y \in S'. \end{cases}$$

One readily checks that $(S \sqcup S', \nu)$ is a semigroup; moreover, (S, μ) is a subsemigroup and (S', μ') is a two-sided ideal. We have $E(S \sqcup S') = E(S) \sqcup E(S')$.

When S (resp. S') has a zero element 0_S (resp. $0_{S'}$), consider the set $S \cup_0 S'$ obtained from $S \sqcup S'$ by identifying 0_S and $0_{S'}$. One checks that $S \cup_0 S'$ has a unique semigroup law ν_0 such that the natural map $S \sqcup S' \rightarrow S \cup_0 S'$ is a homomorphism; moreover, the image of 0_S is the zero element. Here again, S is a subsemigroup of $S \cup_0 S'$, and S' is a two-sided ideal; we have $E(S \cup_0 S') = E(S) \cup_0 E(S')$.

If in addition (S, μ) and (S', μ') are algebraic semigroups, then so are $(S \sqcup S', \nu)$ and $(S \cup_0 S', \nu_0)$. This construction still makes sense when (say) S' is a scheme of finite type over k , equipped with a closed point $0 = 0_{S'}$ and with the associated trivial semigroup law μ_0 . Taking for S' the spectrum of a local ring of finite dimension as a k -vector space, and for 0 the unique closed point of S' , we obtain many examples of nonreduced semigroup schemes (having a fat point at their zero element).

(iv) Any finite semigroup is algebraic. In the opposite direction, the (finite) set of connected components of an algebraic semigroup (S, μ) has a natural structure of semigroup. Indeed, if C_1, C_2 are connected components of S , then $\mu(C_1, C_2)$ is contained in a unique connected component, $C_1 C_2$. The resulting composition law on the set of connected components, $\pi_0(S)$, is clearly associative, and the canonical map $f : S \rightarrow \pi_0(S)$ is a homomorphism of algebraic semigroups. In fact, f is the universal homomorphism from S to a finite semigroup.

Next, we present examples of algebraic monoids and of algebraic groups:

Examples 2.1.9. (i) Consider the set M_n of $n \times n$ matrices with coefficients in k , where n is a positive integer. We may view M_n as an affine space of dimension n^2 ; this is an irreducible algebraic monoid relative to matrix multiplication, the neutral element being of course the identity matrix.

The subspaces D_n of diagonal matrices, and T_n of upper triangular matrices, are closed irreducible submonoids of M_n . Note that D_n is isomorphic to the affine n -space \mathbb{A}^n equipped with pointwise multiplication.

An example of a closed reducible submonoid of M_n consists of the matrices having at most one nonzero entry in each row and each column. This submonoid, that we denote by R_n , is the closure in M_n of the group of monomial matrices (those having exactly one nonzero entry in each row and each column). Note that $R_n = D_n S_n$, where S_n denotes the symmetric group on n letters, viewed as the group of permutation matrices. Thus, the irreducible components of R_n are parametrized by S_n . Each such component contains the zero matrix; in particular, R_n is connected.

(ii) Let A be a k -algebra. Then the set of algebra endomorphisms of A , equipped with the composition, is an (abstract) monoid that we denote by $\text{End}(A)$. If A is finite-dimensional as a k -vector space, then $\text{End}(A)$ is an algebraic monoid; indeed, it identifies to a closed submonoid of M_n , where $n := \dim(A)$.

(iii) A *linear* algebraic monoid is a closed submonoid M of some matrix monoid M_n . Then the variety M is affine; conversely, every affine algebraic monoid is linear (see [DG70, Thm. II.2.3.3]). It follows that every affine algebraic semigroup is linear as well, see [Pu88, Cor. 3.16].

(iv) Examples of algebraic groups include:

- the *additive group* \mathbb{G}_a , i.e., the affine line equipped with the addition,
- the *multiplicative group* \mathbb{G}_m , i.e., the affine line minus the origin, equipped with the multiplication,
- the *elliptic curves*, i.e., the complete nonsingular irreducible curves of genus 1, equipped with a base point; then there is a unique algebraic group structure for which this point is the neutral element, see e.g. [Ha77, Chap. II, §4].

In fact, these examples yield all the connected algebraic groups of dimension 1, see [Ke93, Prop. 10.7.1].

(v) A complete connected algebraic group is called an *abelian variety*; elliptic curves are examples of such algebraic groups. It is known that every abelian variety A is a commutative group and a projective variety; moreover, the group law on A is uniquely determined by the structure of variety and the neutral element (see [Mu74, Chap. II]).

2.2 The unit group of an algebraic monoid

In this section, we obtain some fundamental results on the group of invertible elements of an algebraic monoid. We shall need the following observation:

Proposition 2.2.1. *Let (M, μ) be an algebraic monoid. Then M has a unique irreducible component containing 1: the neutral component M° . Moreover, $M^\circ X = X M^\circ = X$ for any irreducible component X of M ; in particular, M° is a closed submonoid of M .*

Proof. Let X, Y be irreducible components of M . Then XY is the image of the restriction of μ to $X \times Y$, and hence is a constructible subset of M ; moreover, its closure \overline{XY} is an irreducible subvariety of M . If $1 \in X$, then $Y \subseteq XY \subseteq \overline{XY}$. Since Y is an irreducible component, we must have $Y = XY = \overline{XY}$; likewise, one obtains that $YX = Y$. In particular, $XX = X$, i.e., X is a closed submonoid. If in addition $1 \in Y$, then also $XY = YX = Y$, hence $Y = X$. This yields our assertions. \square

Remark 2.2.2. Any algebraic group G is a nonsingular variety, and hence every connected component of G is irreducible. Moreover, the neutral component G^o is a closed normal subgroup, and the quotient group G/G^o parametrizes the components of G .

In contrast, there exist connected reducible algebraic monoids: for example, the monoid R_n of Example 2.1.9 (i). Also, algebraic monoids are generally singular; e.g., the zero locus of $z^2 - xy$ in \mathbb{A}^3 equipped with pointwise multiplication.

On a more advanced level, note that any group scheme is reduced in characteristic 0 (see e.g. [DG70, Thm. II.6.1.1]). In contrast, there always exist nonreduced monoid schemes. For example, one may stick an arbitrary fat point at the origin of the multiplicative monoid (\mathbb{A}^1, \times) , by the construction of Example 2.1.8 (iii).

Definition 2.2.3. Let M be a monoid and let $x, y \in M$. Then y is a *left* (resp. *right*) *inverse* of x if $yx = 1$ (resp. $xy = 1$). We say that x is *invertible* (also called a *unit*) if it has a left and a right inverse.

With the above notation, one readily checks that the left and right inverses of any unit $x \in M$ are equal. Moreover, if $x' \in M$ is another unit with inverse y' , then xy' is a unit with inverse $x'y$. Thus, the invertible elements of M form a subgroup: the *unit group*, that we denote by $G(M)$.

The following result on unit groups of algebraic monoids is due to Rittatore in the irreducible case (see [Ri98, Thm. 1]). The proof presented here follows similar arguments.

Theorem 2.2.4. *Let M be an algebraic monoid. Then $G(M)$ is an algebraic group, open in M . In particular, $G(M)$ consists of nonsingular points of M .*

Proof. Let

$$G := \{(x, y) \in M \times M^{\text{op}} \mid xy = yx = 1\},$$

where M^{op} denotes the *opposite* monoid to M , i.e., the variety M equipped with the composition law $(x, y) \mapsto yx$. One readily checks that G (viewed as a closed subvariety of $M \times M^{\text{op}}$) is a submonoid; moreover, every $(x, y) \in G$ has inverse (y, x) . Thus, G is a closed algebraic subgroup of $M \times M^{\text{op}}$.

The first projection $p_1 : M \times M^{\text{op}} \rightarrow M$ restricts to a homomorphism of monoids $\pi : G \rightarrow M$ with image being the unit group $G(M)$. In fact, G acts on M by left multiplication: $(x, y) \cdot z := xz$, and π is the orbit map $(x, y) \mapsto (x, y) \cdot 1$; in particular, $G(M)$ is the G -orbit of 1. The isotropy subgroup of 1 in G is clearly trivial as a set. We claim that this also holds as a scheme; in other words, the isotropy Lie algebra of 1 is trivial as well.

To check this, recall that the Lie algebra $\text{Lie}(G)$ is the Zariski tangent space $T_{(1,1)}(G)$ and hence is contained in $T_{(1,1)}(M \times M^{\text{op}}) \cong T_1(M) \times T_1(M)$. Since the differential at $(1, 1)$ of the monoid law $\mu : M \times M \rightarrow M$ is the map

$$T_1(M) \times T_1(M) \longrightarrow T_1(M), \quad (x, y) \longmapsto x + y,$$

we have

$$T_{(1,1)}(G) \subseteq \{(x, y) \in T_1(M) \times T_1(M) \mid x + y = 0\}.$$

Thus, the first projection $\text{Lie}(G) \rightarrow T_1(M)$ is injective; but this projection is the differential of π at $(1, 1)$. This proves our claim.

By that claim, π is a locally closed immersion. Thus, $G(M)$ is a locally closed subvariety of M , and π induces an isomorphism of groups $G \cong G(M)$. So $G(M)$ is an algebraic group.

It remains to show that $G(M)$ is open in M ; it suffices to check that $G(M)$ contains an open subset U of M (then the translates gU , where $g \in G(M)$, form a covering of $G(M)$ by open subsets of M). For this, we may replace M with its neutral component M^o (Proposition 2.2.1) and hence assume that M is irreducible. Note that

$$G(M) = \{x \in M \mid xy = zx = 1 \text{ for some } y, z \in M\}$$

(then $y = zxy = z$). In other words,

$$G(M) = p_1(\mu^{-1}(1)) \cap p_2(\mu^{-1}(1)),$$

where $p_1, p_2 : M \times M \rightarrow M$ denote the projections. Also, the set-theoretic fiber at 1 of the restriction $p_1 : \mu^{-1}(1) \rightarrow M$ consists of the single point 1. By a classical result on the dimension of fibers of a morphism (see [Ha77, Exer. II.3.22]), it follows that every irreducible component C of $\mu^{-1}(1)$ containing 1 satisfies $\dim(C) = \dim(M)$, and the restriction $p_1 : C \rightarrow M$ is dominant. Thus, $p_1(C)$ contains a dense open subset of M . Likewise, $p_2(C)$ contains a dense open subset of M , and hence so does G . \square

Note that the unit group of a linear algebraic group is linear, see [Pu88, Cor. 3.26]. Further properties of the unit group are gathered in the following:

Proposition 2.2.5. *Let M be an algebraic monoid, and G its unit group.*

- (i) *If $x \in M$ has a left (resp. right) inverse, then $x \in G$.*
- (ii) *$M \setminus G$ is the smallest proper two-sided ideal of M .*
- (iii) *If 1 is the unique idempotent of M , then $M = G$.*

Proof. (i) Assume that x has a left inverse y . Then the left multiplication $M \rightarrow M$, $z \mapsto xz$ is an injective endomorphism of the variety M . By [Ax68, Thm. C] (see also [Bo69]), this endomorphism is surjective, and hence there exists $z \in M$ such that $xz = 1$. Then $y = yxz = z$, i.e., $x \in G$. The case where x has a right inverse is handled similarly.

(ii) Clearly, any proper two-sided ideal of M is contained in $M \setminus G$. We show that the latter is a two-sided ideal: let $x \in M \setminus G$ and $y \in M$. If $xy \in G$, then $y(xy)^{-1}$ is a right inverse of x . By (i), it follows that $x \in G$, a contradiction. Thus, $(M \setminus G)M \subseteq M \setminus G$. Likewise, $M(M \setminus G) \subseteq M \setminus G$.

(iii) By Theorem 2.2.4, $M \setminus G$ is closed in M ; also, $M \setminus G$ is a subsemigroup of M by (ii). Thus, if $M \neq G$ then $M \setminus G$ contains an idempotent, in view of Proposition 2.1.6. \square

2.3 The kernel of an algebraic semigroup

In this subsection, we show that every algebraic semigroup has a smallest two-sided ideal (called its *kernel*) and we describe the structure of that ideal, thereby generalizing some of the known results about the kernel of a linear algebraic semigroup (see [Pu88, Hu05]).

First, recall that the idempotents of any (abstract) semigroup S are in bijective correspondence with the maximal submonoids of S via $e \mapsto eSe$, and hence with the maximal

subgroups of S via $e \mapsto G(eSe)$. Thus, when S is an algebraic semigroup, its maximal subgroups are all locally closed, in view of Theorem 2.2.4.

Next, we recall the classical definition of a partial order on the set of idempotents of any (abstract) semigroup:

Definition 2.3.1. Let S be a semigroup and let $e, f \in E(S)$. Then $e \leq f$ if we have $e = ef = fe$.

Note that $e \leq f$ if and only if $e \in fSf$; this is also equivalent to the condition that $eSe \subseteq fSf$. Thus, \leq is indeed a partial order on $E(S)$ (this fact may of course be checked directly). Also, note that \leq is preserved by every homomorphism of semigroups. For an algebraic semigroup, the partial order \leq satisfies additional finiteness properties:

Proposition 2.3.2. *Let S be an algebraic semigroup.*

- (i) *Every subset of $E(S)$ has a minimal element with respect to the partial order \leq , and also a maximal element.*
- (ii) *$e \in E(S)$ is minimal among all idempotents if and only if eSe is a group.*
- (iii) *If S is commutative, then $E(S)$ is finite and has a smallest element.*

Proof. (i) Note that the eSe , where $e \in E(S)$, form a family of closed subsets of the noetherian topological space S ; hence any subfamily has a minimal element. For the existence of maximal elements, consider the family

$$S \times_{Se} S := \{(x, y) \in S \times S \mid xe = ye\}$$

of closed subsets of $S \times S$. If $e \leq f$, then $S \times_{Sf} S \subseteq S \times_{Se} S$, since the equality $xf = yf$ implies that $xe = xfe = yfe = ye$. Moreover, if $e \leq f$ and $S \times_{Se} S = S \times_{Sf} S$, then $xe = ye \Leftrightarrow xf = yf$ for $x, y \in S$. In particular, xf only depends on xe , and hence $xf = xef$. So $f = fef = e$. Thus, a minimal $S \times_{Se} S$ (for e in a given subset of $E(S)$) yields a maximal e .

(ii) Let e be a minimal idempotent of S . Then e is the unique idempotent of the algebraic monoid eSe . By Proposition 2.2.5 (iii), it follows that eSe is a group. The converse is immediate (and holds for any abstract semigroup).

(iii) Let e, f be idempotents. Then $ef = fe$ is also idempotent, and $ef = e(ef)e = f(ef)f$ so that $ef \leq e$ and $ef \leq f$. Thus, any two minimal idempotents are equal, i.e., $E(S)$ has a smallest element.

To show that $E(S)$ is finite, we may replace S with its closed subsemigroup $E(S)$, and hence assume that every element of S is idempotent. Then every connected component of S is a closed subsemigroup in view of Example 2.1.8 (iv). So we may further assume that S is connected. Let $x \in S$; then xS is a connected commutative algebraic monoid with neutral element x , and consists of idempotents. Thus, $G(xS) = \{x\}$. By Theorem 2.2.4, it follows that x is an isolated point of xS . Hence $xS = \{x\}$, i.e., $xy = x$ for all $y \in S$. Since S is commutative, we must have $S = \{x\}$. \square

As a consequence of the above proposition, every algebraic semigroup admits minimal idempotents. These are of special interest, as shown by the following:

Proposition 2.3.3. *Let S be an algebraic semigroup, $e \in S$ a minimal idempotent, and eSe the associated closed subgroup of S .*

(i) *The map*

$$\rho : S \longrightarrow S, \quad s \longmapsto s(ese)^{-1}s$$

is a retraction of varieties of S to SeS . In particular, SeS is a closed two-sided ideal of S .

(ii) *The map*

$$\varphi : {}_eSe \times {}_eSe \times {}_eSe \longrightarrow S, \quad (x, g, y) \longmapsto xgy$$

yields an isomorphism of varieties to its image, SeS .

(iii) *Via the above isomorphism, the semigroup law on SeS is identified to that on ${}_eSe \times {}_eSe \times {}_eSe$ given by*

$$(x, g, y)(x', g', y') = (x, g\pi(y, x')g', y'),$$

where $\pi : {}_eSe \times {}_eSe \rightarrow {}_eSe$ denotes the map $(y, z) \mapsto yz$. This identifies the idempotents of SeS to the triples $(x, \pi(y, x)^{-1}, y)$, where $x \in {}_eSe$ and $y \in {}_eSe$. In particular, $E(SeS) \cong {}_eSe \times {}_eSe$ as a variety.

(iv) *The semigroup SeS has no proper two-sided ideal.*

(v) *SeS is the smallest two-sided ideal of S ; in particular, it does not depend on the minimal idempotent e .*

(vi) *The minimal idempotents of S are exactly the idempotents of SeS .*

Proof. (i) Clearly, ρ is a morphism; also, since $s(ese)^{-1}s \in SeSeS$ for all $s \in S$, the image of ρ is contained in SeS . Let $s, t \in S$; then

$$set = se(ese)^{-1}esete(ete)^{-1}t = xgy,$$

where $x := se(ese)^{-1}$, $g := esete$ and $y := (ete)^{-1}et$. Moreover, $x \in {}_eSe$, $g \in {}_eSe$ and $y \in {}_eSe$. In particular, φ is surjective; also,

$$\rho(set) = xgyg^{-1}xgy = xgyeg^{-1}exgy = xgeg^{-1}egy = xgy = set.$$

Hence ρ is a retraction of S to SeS ; in particular, SeS is closed in S .

(ii) The map

$$\psi : SeS \longrightarrow {}_eSe \times {}_eSe \times {}_eSe, \quad set \longmapsto (x, g, y)$$

(where x, g, y are defined as above) is a morphism of varieties and satisfies $\varphi \circ \psi = \text{id}$. Thus, it suffices to check that $\psi \circ \varphi = \text{id}$. Let $x \in {}_eSe$, $g \in {}_eSe$, $y \in {}_eSe$ and put $s := xgy$. Then $se = xg$ and $es = gy$. Hence $g = ese$, $x = se(ese)^{-1}$ and $y = (ese)^{-1}es$, which yields the desired assertion.

(iii) For the first assertion, just write $(xgy)(x'g'y') = x(g(yx')g')y'$, and note that $yx' \in {}_eSe {}_eSe \subseteq {}_eSe$. The assertions on idempotents follow readily.

(iv) Let $s \in SeS$ and write $s = \varphi(x, g, y)$. Then the subset SsS of S is identified with that of ${}_eSe \times {}_eSe \times {}_eSe$ consisting of the triples $(x_1, g_1\pi(y_1, x)g\pi(y, x_2)g_2, y_2)$, where $x_1, x_2 \in {}_eSe$, $g_1, g_2 \in {}_eSe$ and $y_1, y_2 \in {}_eSe$. It follows that $SsS = SeS$, and hence that SeS is the smallest two-sided ideal containing s .

(v) Let I be a two-sided ideal of S . Then SeI is a two-sided ideal of S contained in SeS ; hence $SeI = SeS$ by (iv). But $SeI \subseteq I$; this yields our assertions.

(vi) If $f \in E(S)$ is minimal, then $SfS = SeS$ by (v). In particular, $f \in SeS$.

For the converse, let $f \in E(SeS)$. Then $SfS = SeS$ by (iv), and hence $fSf = fSfSf = f(SeS)f$. Identifying f to a triple $(x, \pi(y, x)^{-1}, y)$, one checks as in the proof of (iv) that $f(SeS)f$ is identified to the set of triples (x, g, y) , where $g \in eSe$. But $(x, \pi(y, x)^{-1}, y)$ is the unique idempotent of this set. Thus, f is the unique idempotent of fSf , i.e., f is minimal. \square

In view of these results, we may set up the following:

Definition 2.3.4. The *kernel* of an algebraic semigroup S is the smallest two-sided ideal of S , denoted by $\ker(S)$.

Remarks 2.3.5. (i) As a consequence of Proposition 2.3.3, we see that any algebraic semigroup having no proper closed two-sided ideal is *simple*, i.e., has no proper two-sided ideal at all. Moreover, any simple algebraic semigroup S , equipped with an idempotent e , is isomorphic (as a variety) to the product $X \times G \times Y$, where $X := {}_eSe$ and $Y := eS_e$ are varieties, and $G := eSe$ is an algebraic group. This identifies e to a point of the form $(x_o, 1, y_o)$, where $x_o \in X$ and $y_o \in Y$; moreover, the semigroup law of S is identified to that as in Proposition 2.3.3 (iii), where $\pi : Y \times X \rightarrow G$ is a morphism such that $\pi(x_o, y) = \pi(x, y_o) = 1$ for all $x \in X$ and $y \in Y$.

Conversely, any tuple (X, Y, G, π, x_o, y_o) satisfying the above conditions defines a semigroup law on $S := X \times G \times Y$ such that $e := (x_o, 1, y_o)$ is idempotent and ${}_eSe = X \times \{(1, y_o)\}$, $eSe = \{x_o\} \times G \times \{y_o\}$ and $eS_e = \{(x_o, 1)\} \times Y$.

This description of algebraic semigroups having no proper closed two-sided ideal is a variant of the classical Rees structure theorem for those (abstract) semigroups that are *completely simple*, i.e., simple and having a minimal idempotent (see e.g. [Pu88, Thm. 1.9]).

(ii) By analogous arguments, one shows that every algebraic semigroup S contains minimal left ideals, and these are exactly the subsets Sf , where f is a minimal idempotent. In particular, the minimal left ideals are all closed. Also, given a minimal idempotent e , these ideals are exactly the subsets $X \times G \times \{y\}$ of $\ker(S)$, where $X := {}_eSe$, $G := eSe$ and $y \in Y := eS_e$ as above. Similar assertions hold of course for the minimal right ideals; it follows that the intersections of minimal left and minimal right ideals are exactly the subsets $\{x\} \times G \times \{y\}$, where $x \in X$ and $y \in Y$.

2.4 Unit dense algebraic monoids

In this subsection, we introduce and study the class of unit dense algebraic monoids. These include the irreducible algebraic monoids, and will play an important role in their structure.

Let M be an algebraic monoid, and $G(M)$ its unit group. Then the algebraic group $G(M) \times G(M)$ acts on M via left and right multiplication: $(g, h) \cdot x := gxh^{-1}$. Moreover, the orbit of 1 under this action is just $G(M)$, and the isotropy subgroup scheme of 1 is the diagonal, $\text{diag}(G(M)) := \{(g, g) \mid g \in G(M)\}$.

Definition 2.4.1. An algebraic monoid M is *unit dense* if $G(M)$ is dense in M .

For instance, every irreducible algebraic monoid is unit dense. An example of a reducible unit dense algebraic monoid consists of the $n \times n$ matrices having at most one nonzero entry in each row and each column (Example 2.1.9 (i)).

Any unit dense monoid may be viewed as an equivariant embedding of its unit group, in the sense of the following:

Definition 2.4.2. Let G be an algebraic group. An *equivariant embedding* of G is a variety X equipped with an action of $G \times G$ and with a point $x \in X$ such that the orbit $(G \times G) \cdot x$ is dense in X , and the isotropy subgroup scheme $(G \times G)_x$ is the diagonal $\text{diag}(G)$.

Note that the law of a unit dense monoid is uniquely determined by its structure of equivariant embedding, since that structure yields the law of the unit group. Also, given an *affine* algebraic group G , every *affine* equivariant embedding of G has a unique structure of algebraic monoid such that G is the unit group, see [Ri98, Prop. 1]; conversely, every unit dense algebraic monoid with unit group G is affine by Theorem 3.1.1. For an arbitrary (connected) algebraic group G , the equivariant embeddings of G that admit a monoid structure are characterized in Theorem 3.4.2.

Proposition 2.4.3. *Let M be an algebraic monoid, and G its unit group. Then the unit group of the neutral component M° is the neutral component G° of G .*

If M is unit dense, then its irreducible components are exactly the subsets gM° , where $g \in G$; they are indexed by G/G° , the group of components of G .

Proof. Note that $G(M^\circ)$ is contained in G , and open in M° by Theorem 2.2.4. Hence $G(M^\circ)$ contains an open neighborhood of 1 in M° , or equivalently in G . Using the group structure, it follows that $G(M^\circ)$ is open in G ; also, $G(M^\circ)$ is irreducible since so is M° . But the algebraic group G contains a unique open irreducible subgroup: its neutral component. Thus, $G(M^\circ) = G^\circ$.

Clearly, gM° is an irreducible component of M for any $g \in G$, and this component depends only on the coset gG° . If M is unit dense, then any irreducible component X of M contains a unit, say g . Since $g^{-1}X$ is an irreducible component containing 1, it follows that $X = gM^\circ$. If $X = hM^\circ$ for some $h \in G$, then $g^{-1}h \in G \cap M^\circ$. Thus, $g^{-1}hG^\circ$ is an open subset of M° , and hence meets G° ; so $g^{-1}h \in G^\circ$, i.e., $gG^\circ = hG^\circ$. \square

Proposition 2.4.4. *Let M be a unit dense algebraic monoid, and G its unit group. Then the kernel, $\ker(M)$, is the unique closed orbit of $G \times G$ acting by left and right multiplication. Moreover, $\ker(M) = GeG$ for any minimal idempotent e of M .*

Proof. We may choose a closed $G \times G$ -orbit Y in M . Then

$$MYM = \overline{GYG} \subseteq \overline{GYG} = \overline{Y} = Y.$$

Thus, Y is a two-sided ideal of M . Moreover, if Z is another two-sided ideal, then Z is stable by $G \times G$, and meets Y since $YZ \subseteq Y \cap Z$. Thus, Z contains Y ; this shows that $Y = \ker(M)$. In particular, Y is the unique closed $G \times G$ -orbit; this proves the first assertion. The second one follows from Proposition 2.3.3. \square

Proposition 2.4.5. *Let M be a unit dense algebraic monoid with unit group G . Then the following conditions are equivalent for any $x \in M$:*

- (i) *The orbit Gx (for the G -action on M by left multiplication) is closed in M .*
- (ii) *$Gx = Mx$.*
- (iii) *$x \in \ker(M)$.*

Moreover, all closed G -orbits in M are equivariantly isomorphic; in other words, the isotropy subgroup schemes G_x , where $x \in \ker(M)$, are all conjugate. Also, each closed orbit contains a minimal idempotent. For any such idempotent e , the algebraic group eMe equals eGe .

Proof. (i) \Rightarrow (ii) Since Gx is closed in M , we have $Mx = \overline{Gx} \subseteq \overline{Gx} = Gx$ and hence $Mx = Gx$.

(ii) \Rightarrow (iii) We have $Gx = Mx \supset \ker(M)x$ and the latter subset is stable under left multiplication by G . Hence $Gx = \ker(M)x$ is contained in $\ker(M)$.

(iii) \Rightarrow (i) Let e be a minimal idempotent of M . Since $\ker(M) = GeG$, the G -orbits in $\ker(M)$ are exactly the orbits Geg , where $g \in G$. Since the right multiplication by g is an automorphism of the variety M commuting with left multiplications, these orbits are all isomorphic as G -varieties. In particular, they all have the same dimension; hence they are closed in $\ker(M)$, and thus in M . Also, the orbit Geg contains $g^{-1}eg$, which is a minimal idempotent since the map $M \rightarrow M$, $x \mapsto g^{-1}xg$ is an automorphism of algebraic monoids. Finally, we have $Ge = Me$ by (i); likewise, $eG = eM$ and hence $eGe = eMe$. \square

Note that the closed orbits for the left G -action are exactly the minimal left ideals (considered in Remark 2.3.5 (ii) in the setting of algebraic semigroups).

2.5 The normalization of an algebraic semigroup

In this subsection, we begin by recalling some background results on the normalization of an arbitrary variety (see e.g. [Ei95, §4.2, §11.2]). Then we discuss the normalization of algebraic semigroups and monoids; as in the previous subsection, this construction will play an important role in their structure.

A variety X is *normal* at a point x if the local ring $\mathcal{O}_{X,x}$ is integrally closed in its total quotient ring; X is normal if it is so at any point. The normal points of a variety form an open subset, which contains the nonsingular points. The irreducible components of a normal variety are pairwise disjoint, and each of them is normal.

An arbitrary variety X has a *normalization*, i.e., a normal variety \tilde{X} together with a finite surjective morphism $\eta : \tilde{X} \rightarrow X$ which satisfies the following universal property: for any normal variety Y and any morphism $\varphi : Y \rightarrow X$ having a dense image, there exists a unique morphism $\tilde{\varphi} : Y \rightarrow \tilde{X}$ such that $\varphi = \eta \circ \tilde{\varphi}$. Then \tilde{X} is uniquely determined up to unique isomorphism, and η is an isomorphism above the open subset of normal points of X ; in particular, η is birational (i.e., an isomorphism over a dense open subset of X).

Proposition 2.5.1. *Let (S, μ) be an algebraic semigroup and let $\eta : \tilde{S} \rightarrow S$ be the normalization.*

(i) *If the morphism $\mu : S \times S \rightarrow S$ is dominant, then \tilde{S} has a unique algebraic semigroup law $\tilde{\mu}$ such that η is a homomorphism. Moreover, $\eta(E(\tilde{S})) = E(S)$.*

(ii) If S is an algebraic monoid (so that μ is surjective), then \tilde{S} is an algebraic monoid as well, with neutral element the unique preimage of 1_S under η . Moreover, η induces an isomorphism $G(\tilde{S}) \cong G(S)$.

Proof. (i) By the assumption on μ , the morphism $\mu \circ (\eta \times \eta) : \tilde{S} \times \tilde{S} \rightarrow S$ is dominant. Since $\tilde{S} \times \tilde{S}$ is normal, there exists a unique morphism $\tilde{\mu} : \tilde{S} \times \tilde{S} \rightarrow \tilde{S}$ such that $\eta \circ \tilde{\mu} = \mu \circ (\eta \times \eta)$. Then $\tilde{\mu}$ is associative, since it coincides with μ on the dense open subset of normal points; moreover, η is a homomorphism by construction. The assertion on idempotents is a consequence of Corollary 2.1.7.

(ii) The neutral element 1_S is a nonsingular point of S by Theorem 2.2.4; thus, it has a unique preimage $1_{\tilde{S}}$ under η . Moreover, we have for any $\tilde{x} \in \tilde{S}$:

$$\eta(\tilde{\mu}(\tilde{x}, 1_{\tilde{S}})) = \mu(\eta(\tilde{x}), \eta(1_{\tilde{S}})) = \eta(\tilde{x}) = \eta(\tilde{\mu}(1_{\tilde{S}}, \tilde{x})).$$

Thus, $\tilde{\mu}(\tilde{x}, 1_{\tilde{S}}) = \tilde{\mu}(1_{\tilde{S}}, \tilde{x}) = \tilde{x}$ for all \tilde{x} such that $\eta(\tilde{x})$ is a normal point of S . By density of these points, it follows that $1_{\tilde{S}}$ is the neutral element of $(\tilde{S}, \tilde{\mu})$. Finally, the assertion on unit groups follows from the inclusion $G(\tilde{S}) \subseteq \eta^{-1}(G(S))$ and from the fact that η is an isomorphism above the nonsingular locus of S . \square

Remarks 2.5.2. (i) For an arbitrary algebraic semigroup S , there may exist several algebraic semigroup laws on the normalization \tilde{S} that lift μ . For example, let $x \in S$ and consider the trivial semigroup law μ_x of Example 2.1.8 (i). Then $\mu_{\tilde{x}}$ lifts μ for any $\tilde{x} \in \tilde{X}$ such that $\eta(\tilde{x}) = x$. In general, such a point \tilde{x} is not unique, e.g., when S is a plane curve and x an ordinary multiple point.

(ii) With the above notation, there may also exist no algebraic semigroup law on \tilde{S} that lifts μ . To construct examples of such algebraic semigroups, consider a normal irreducible affine variety X and a complete nonsingular irreducible curve C . Then we may choose a finite surjective morphism $\varphi : C \rightarrow \mathbb{P}^1$. Let $Y := C \setminus \{\varphi^{-1}(\infty)\}$; this is an affine nonsingular irreducible curve equipped with a finite surjective morphism $\varphi : Y \rightarrow \mathbb{A}^1$. Choose a point $x_o \in X$ and let $\gamma : Y \rightarrow X \times Y$, $y \mapsto (x_o, y)$; then γ is a section of the second projection $p_2 : X \times Y \rightarrow Y$. By [Fe03, Thm. 5.1], there exists a unique irreducible variety S that sits in a co-cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\ \gamma \downarrow & & \downarrow \iota \\ X \times Y & \xrightarrow{\eta} & S. \end{array}$$

Then ι is a closed immersion, and η is a finite morphism that restricts to an isomorphism $(X \setminus \{x_o\}) \times Y \cong S \setminus \iota(\mathbb{A}^1)$ and to the morphism $\{x_o\} \times Y \rightarrow \mathbb{A}^1$, $(x_o, y) \mapsto \varphi(y)$. In particular, η is the normalization; S is obtained by “pinching $X \times Y$ along $\{x_o\} \times Y$ via φ ”. Since the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & \mathbb{A}^1 \\ \gamma \downarrow & & \downarrow \text{id} \\ X \times Y & \xrightarrow{\varphi \circ p_2} & \mathbb{A}^1 \end{array}$$

commutes, it yields a unique morphism $\rho : S \rightarrow \mathbb{A}^1$ such that $\rho \circ \varphi = \text{id}$. The retraction ρ defines in turn an algebraic semigroup law μ on S by $\mu(s, s') := \rho(s)\rho(s')$ as in Example 2.1.8 (ii).

We claim that μ does not lift to any algebraic semigroup law on $X \times Y$, if the curve C is nonrational. Indeed, any such lift $\tilde{\mu}$ satisfies

$$\eta(\tilde{\mu}((x, y), (x', y'))) = \mu(\eta(x, y), \eta(x', y')) = \iota(\rho(\eta(x, y), \eta(x', y'))) = \iota(\varphi(y)\varphi(y'))$$

for any $x, x' \in X$ and any $y, y' \in Y$. As a consequence, $\tilde{\mu}((x, y), (x', y'))$ only depends on (y, y') , and this yields an algebraic semigroup law on Y such that ρ is a homomorphism. But such a law does not exist, as follows e.g. from Theorem 4.2.1.

3 Irreducible algebraic monoids

3.1 Algebraic monoids with affine unit group

The aim of this subsection is to prove the following result, due to Rittatore for irreducible algebraic monoids (see [Ri07, Thm. 5]). The proof presented here follows his argument closely, except for an intermediate step (Proposition 3.1.2).

Theorem 3.1.1. *Let M be a unit dense algebraic monoid, and G its unit group. If G is affine, then so is M .*

Proof. Let $\eta : \tilde{M} \rightarrow M$ denote the normalization. Then \tilde{M} is an algebraic monoid with unit group isomorphic to G , by Proposition 2.5.1. Moreover, G is dense in \tilde{M} since it is so in M . If \tilde{M} is affine, then M is affine by a result of Chevalley: the image of an affine variety by a finite morphism is affine (see [Ha77, Exer. II.4.2]). Thus, we may assume that M is normal. Then M is the disjoint union of its irreducible components, and each of them is isomorphic (as a variety) to the neutral component M° (Proposition 2.4.3). So we may assume in addition that M is irreducible.

By Proposition 2.4.4, the connected algebraic group $G \times G$ acts on M with a unique closed orbit. In view of a result of Sumihiro (see [Su74]), it follows that M is quasiprojective; in other words, there exists a locally closed immersion $\iota : X \rightarrow \mathbb{P}^n$ for some positive integer n . (We may further assume that ι is equivariant for some linear action of G on \mathbb{P}^n ; we will not need that fact in this proof). Then the pull-back $L := \iota^* \mathcal{O}_{\mathbb{P}^n}(1)$ is an ample line bundle on M . The associated principal \mathbb{G}_m -bundle $\pi : X \rightarrow M$ (where X is the complement of the zero section in L) is the pull-back to M of the standard principal \mathbb{G}_m -bundle $\mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. Thus, X is a locally closed subvariety of \mathbb{A}^{n+1} , and hence is quasi-affine.

By Proposition 3.1.2 below (a version of [Ri07, Thm. 4]), X has a structure of algebraic monoid such that π is a homomorphism. Since that monoid is quasi-affine, it is in fact affine by a result of Renner (see [Re84, Thm. 4.4]). Moreover, the map $\pi : X \rightarrow M$ is the categorical quotient by the action of \mathbb{G}_m ; hence M is affine. \square

Proposition 3.1.2. *Let M be a normal irreducible algebraic monoid, and assume that its unit group G is affine. Let $\varphi : L \rightarrow M$ be a line bundle, and $\pi : X \rightarrow M$ the associated principal \mathbb{G}_m -bundle. Then X has a structure of a normal irreducible algebraic monoid such that π is a homomorphism.*

Proof. By [KKV89, Lem. 4.3], the preimage $Y := \pi^{-1}(G)$ has a structure of algebraic group such that the restriction of π is a homomorphism; we then have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow Y \xrightarrow{\pi} G \longrightarrow 1,$$

where \mathbb{G}_m is contained in the center of Y . Thus, the group law $\mu_G : G \times G \rightarrow G$ sits in a cartesian square

$$\begin{array}{ccc} Y \times^{\mathbb{G}_m} Y & \xrightarrow{\mu_Y} & Y \\ \pi \times \pi \downarrow & & \downarrow \pi \\ G \times G & \xrightarrow{\mu_G} & G, \end{array}$$

where $Y \times^{\mathbb{G}_m} Y$ denotes the quotient of $Y \times Y$ by the action of \mathbb{G}_m via $t \cdot (y, z) = (ty, t^{-1}z)$, and μ_Y stands for the group law on Y . Via the correspondence between principal \mathbb{G}_m -bundles and line bundles, this translates into a cartesian square

$$\begin{array}{ccc} p_1^*(L|_G) \otimes p_2^*(L|_G) & \longrightarrow & L|_G \\ \varphi \times \varphi \downarrow & & \downarrow \varphi \\ G \times G & \xrightarrow{\mu_G} & G, \end{array}$$

where $p_1, p_2 : G \times G \rightarrow G$ denote the projections. In other words, we have an isomorphism

$$p_1^*(L|_G) \otimes p_2^*(L|_G) \xrightarrow{\cong} \mu_G^*(L|_G)$$

of line bundles over $G \times G$.

Over $M \times M$, this yields an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L) \otimes \mathcal{O}_{M \times M}(D),$$

where D is a Cartier divisor with support in $(M \times M) \setminus (G \times G)$. Since G is affine, the irreducible components E_1, \dots, E_n of $M \setminus G$ are divisors of M . Thus, the irreducible components of $(M \times M) \setminus (G \times G)$ are exactly the divisors $E_i \times M$ and $M \times E_j$, where $i, j = 1, \dots, n$. Hence $D = p_1^*(D_1) + p_2^*(D_2)$ for some Weil divisors D_1, D_2 with support in $M \setminus G$. We thus have an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L) \otimes p_1^*(\mathcal{O}_M(D_1)) \otimes p_2^*(\mathcal{O}_M(D_2))$$

of line bundles over $M \times M$. We now pull back this isomorphism to $M \times \{1\}$. Note that $\mu^*(L)|_{M \times \{1\}} = L = p_1^*(L)|_{M \times \{1\}}$; also, $p_1^*(\mathcal{O}_M(D_1))|_{M \times \{1\}} = \mathcal{O}_M(D_1)$, and both $p_2^*(L)|_{M \times \{1\}}$, $p_2^*(\mathcal{O}_M(D_2))|_{M \times \{1\}}$ are trivial. Thus, $\mathcal{O}_M(D_1)$ is trivial; one shows similarly that $\mathcal{O}_M(D_2)$ is trivial. Hence we have in fact an isomorphism

$$p_1^*(L) \otimes p_2^*(L) \xrightarrow{\cong} \mu^*(L).$$

As above, this translates into a cartesian square

$$\begin{array}{ccc} X \times^{\mathbb{G}_m} X & \longrightarrow & X \\ \pi \times \pi \downarrow & & \downarrow \pi \\ M \times M & \xrightarrow{\mu} & M. \end{array}$$

In turn, this yields a morphism $\nu : X \times X \rightarrow X$ which lifts $\mu : M \times M \rightarrow M$ and extends the group law $Y \times Y \rightarrow Y$. It follows readily that ν is associative and has 1_Y as a neutral element. \square

3.2 Induction of algebraic monoids

In this subsection, we show that any unit dense algebraic monoid has a universal homomorphism to an algebraic group, and we study the fibers of this homomorphism.

Proposition 3.2.1. *Let M be a unit dense algebraic monoid, and G its unit group.*

(i) *There exists a homomorphism of algebraic monoids $\varphi : M \rightarrow \mathcal{G}(M)$, where $\mathcal{G}(M)$ is an algebraic group, such that every homomorphism of algebraic monoids $\psi : M \rightarrow \mathcal{G}$, where \mathcal{G} is an algebraic group, factors uniquely as φ followed by a homomorphism of algebraic groups $\mathcal{G}(M) \rightarrow \mathcal{G}$.*

(ii) *We have $\mathcal{G}(M) = \varphi(M) = \varphi(G) = G/H$, where H denotes the smallest normal subgroup scheme of G containing the isotropy subgroup scheme G_x for some $x \in \ker(M)$.*

Proof. We show both assertions simultaneously. Let $\psi : M \rightarrow \mathcal{G}$ be a homomorphism as in the statement. Then $\psi|_G$ is a homomorphism of algebraic groups, and hence its image is a closed subgroup of \mathcal{G} . Since M is unit dense, it follows that $\psi(M) = \psi(G)$. Let K be the scheme-theoretic kernel of $\psi|_G$. Then K is a normal subgroup scheme of G , and ψ induces an isomorphism from G/K to $\psi(G)$; we may thus view ψ as a G -equivariant homomorphism $M \rightarrow G/K$. In particular, for any $x \in M$, the map $g \mapsto \psi(g \cdot x)$ yields a morphism $G \rightarrow G/K$ which is equivariant under the action of G by left multiplication, and invariant under the action of the isotropy subgroup scheme G_x by right multiplication. Thus, K contains G_x ; hence K contains H , and $\psi|_G$ factors as the quotient homomorphism $\gamma : G \rightarrow G/H$ followed by the canonical homomorphism $\pi : G/H \rightarrow G/K$.

Next, choose $x \in \ker(M)$; then $Gx = Mx$ by Proposition 2.4.5. Thus, the morphism $M \rightarrow Mx$, $y \mapsto yx$ may be viewed as a morphism $M \rightarrow Gx \cong G/G_x$. Composing with the morphism $G/G_x \rightarrow G/H$ induced by the inclusion of G_x in H , we obtain a morphism $\varphi : M \rightarrow G/H$. Clearly, φ is G -equivariant, and $\varphi(1)$ is the neutral element of G/H . Thus, the restriction $\varphi|_G$ is the quotient homomorphism γ . By density, φ is a homomorphism of monoids, and $\psi = \pi \circ \varphi$. So φ is the desired homomorphism. \square

Remarks 3.2.2. (i) As a consequence of the above proposition, the smallest subgroup scheme of G containing G_x is independent of the choice of $x \in \ker(M)$. This also follows from the fact that the subgroup schemes G_x , where $x \in \ker(M)$, are all conjugate in G (Proposition 2.4.5). By that proposition, we may take for x any minimal idempotent of M .

(ii) As another consequence, any irreducible semigroup S has a universal homomorphism to an algebraic group in the sense of the above proposition. Indeed, choose an idempotent e in S , and consider a homomorphism of semigroups $\psi : S \rightarrow \mathcal{G}$, where \mathcal{G} is an algebraic group. Then $\psi(x) = \psi(exe)$ for all $x \in S$; moreover, eSe is an irreducible monoid with neutral element e . Thus, there exists a unique homomorphism $\pi : \mathcal{G}(eSe) \rightarrow \mathcal{G}$ such that $\psi(x) = \pi(\phi(exe))$ for all $x \in S$, where $\phi : eSe \rightarrow \mathcal{G}(eSe)$ denotes the universal homomorphism. Then we must have $\pi(\phi(exye)) = \pi(\phi(exe)\phi(eye))$

for all $x, y \in S$. Let H denote the smallest normal subgroup scheme of $\mathcal{G}(eSe)$ containing the image of the morphism

$$S \times S \longrightarrow \mathcal{G}(eSe), \quad (x, y) \longmapsto \phi(exye)\phi(exe)^{-1}\phi(eye)^{-1},$$

and let $\varphi : S \rightarrow \mathcal{G}(eSe)/H$ denote the homomorphism that sends every x to the image of exe . Then π factors as φ followed by a unique homomorphism of algebraic groups $\mathcal{G}(eSe)/H \rightarrow \mathcal{G}$, i.e., φ is the desired universal homomorphism. Note that $\varphi : S \rightarrow \mathcal{G}(S)$ is surjective by construction; in particular, \mathcal{G} is connected.

Proposition 3.2.3. *Keep the notation and assumptions of Proposition 3.2.1.*

(i) *If H is an algebraic group (e.g., if $\text{char}(k) = 0$), then the scheme-theoretic fibers of φ are reduced.*

(ii) *If M is normal, then H is a connected algebraic group; moreover, the scheme-theoretic fibers of φ are reduced and irreducible.*

Proof. (i) Denote by $\gamma : G \rightarrow G/H$ the quotient homomorphism and form the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\varphi'} & G \\ \gamma' \downarrow & & \downarrow \gamma \\ M & \xrightarrow{\varphi} & G/H. \end{array}$$

Since γ and φ are equivariant for the actions of G by left multiplication, X is equipped with a G -action such that γ' and φ' are equivariant. Denote by N the (scheme-theoretic) fiber of φ' at the neutral element 1_G . Then the morphism

$$G \times N \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

is an isomorphism with inverse given by $x \mapsto (\varphi'(x), \varphi'(x)^{-1} \cdot x)$. Moreover, the fiber of φ' at every $g \in G$ is $g \cdot N \cong N$. When H is an algebraic group (i.e., when H is smooth; this holds if $\text{char}(k) = 0$), the morphism γ is smooth as well; hence so is γ' . It follows that X is reduced. But $X \cong G \times N$ and hence N is reduced. If in addition H is connected, then the fibers of γ are irreducible; hence the same holds for γ' , and X is irreducible. As above, it follows that N is irreducible.

(ii) Consider the reduced neutral component $H_{\text{red}}^o \subseteq H$; then H_{red}^o is a closed normal subgroup of G . Moreover, the natural map $\delta : G/H_{\text{red}}^o \rightarrow G/H$ is a finite morphism and sits in a commutative square

$$\begin{array}{ccc} G & \xrightarrow{\iota} & M \\ \epsilon \downarrow & & \downarrow \varphi \\ G/H_{\text{red}}^o & \xrightarrow{\delta} & G/H, \end{array}$$

where ι denotes the inclusion. Let $\Gamma \subseteq M \times G/H_{\text{red}}^o$ be the closure of the graph of ϵ . Then the projection $p_1 : \Gamma \rightarrow M$ is a finite morphism, and an isomorphism over the dense open subset G of M . Since M is normal, it follows that p_1 is an isomorphism, i.e., ϵ extends to a morphism $\psi : M \rightarrow G/H_{\text{red}}^o$. As ϵ is a homomorphism of algebraic groups, ψ must be a homomorphism of monoids. Thus, ψ factors through φ , and hence $H_{\text{red}}^o = H$. In other words, H is a connected algebraic group. \square

But in general, the scheme-theoretic fibers of φ are reducible; also, they are nonreduced in prime characteristics, as shown by the following:

Example 3.2.4. Consider the monoid \mathbb{A}^3 equipped with pointwise multiplication, and the locally closed subset

$$M := \{(x, y, z) \mid z^n = xy^n \text{ and } x \neq 0\},$$

where n is a positive integer. Then M is an irreducible commutative algebraic monoid with unit group

$$G = \{(x, y, z) \mid z^n = xy^n \text{ and } z \neq 0\},$$

isomorphic to \mathbb{G}_m^2 via the projection $(x, y, z) \mapsto (y, z)$. Moreover, $\ker(M) = Me$, where $e := (1, 0, 0)$ is the unique minimal idempotent. Since M is commutative, the isotropy subgroup scheme H is just G_e ; the latter is the scheme-theoretic kernel of the homomorphism $x : G \rightarrow \mathbb{G}_m$. Thus,

$$H \cong \{(y, z) \in \mathbb{G}_m^2 \mid y^n = z^n\} \cong \mathbb{G}_m \times \mu_n,$$

where μ_n denotes the subgroup scheme of n th roots of unity. The homomorphism $\varphi : M \rightarrow G/H$ is identified to $x : M \rightarrow \mathbb{G}_m$, and this identifies its fiber at 1 to the submonoid scheme $(y^n = z^n)$ of (\mathbb{A}^2, \times) . The latter scheme is reducible if $n \geq 2$, and nonreduced if n is a multiple of $\text{char}(k)$.

We keep the notation and assumptions of Proposition 3.2.1, and denote by N the scheme-theoretic fiber of φ at 1. Assume in addition that H is an algebraic group (this holds e.g. if M is normal or if $\text{char}(k) = 0$). Then N is reduced by Proposition 3.2.3; also, N is a closed submonoid of M , containing H and stable under the action of G on M by conjugation (via $g \cdot x := gxg^{-1}$). Moreover, the map

$$\pi : G \times N \longrightarrow M, \quad (g, y) \longmapsto gy$$

is a homomorphism of monoids, where $G \times N$ is equipped with the composition law

$$(g_1, y_1)(g_2, y_2) = (g_1g_2, g_2^{-1}y_1g_2y_2)$$

with unit $(1_G, 1_N)$ (this defines the *semi-direct product of G with N*). Finally, π is the quotient morphism for the action of H on $G \times N$ via

$$h \cdot (g, y) := (gh^{-1}, hy).$$

In other words, the morphism $\varphi : M \rightarrow G/H$ identifies M to the fiber bundle

$$\pi : G \times^H N \rightarrow G/H$$

associated to the principal H -bundle $G \rightarrow G/H$ and to the variety N on which H acts by left multiplication. We say that the algebraic monoid M is *induced* from N .

If we no longer assume that H is an algebraic group, then N is just a submonoid scheme of M , and the above properties hold in the setting of monoid schemes. We now obtain slightly weaker versions of these properties in the setting of algebraic monoids.

Proposition 3.2.5. *Let M be a unit dense algebraic monoid, G its unit group, $\varphi : M \rightarrow G/H$ the universal homomorphism to an algebraic group, and N the scheme-theoretic fiber of φ at 1. Denote by H_{red} (resp. N_{red}) the largest closed reduced subscheme of H (resp. N).*

- (i) H_{red} is a closed normal subgroup of G and N_{red} is a closed submonoid of M , stable under the action of G by conjugation.
- (ii) $G \times^{H_{\text{red}}} N_{\text{red}}$ is an algebraic monoid, and the map $\psi : G \times^{H_{\text{red}}} N_{\text{red}} \rightarrow M$ is a finite bijective homomorphism of algebraic monoids.
- (iii) N_{red} is unit dense and its unit group is H_{red} .
- (iv) ψ is birational.

Proof. (i) The assertion on H_{red} is well-known. That on N_{red} follows readily from the fact that N is a closed submonoid scheme of M , stable under the G -action by conjugation.

(ii) The natural map $G/H_{\text{red}} \rightarrow G/H$ is a purely inseparable homomorphism of algebraic groups, and hence is finite and bijective. Also, $G \times^{H_{\text{red}}} N$ is the fibered product of $M = G \times^H N$ and G/H_{red} over G/H . Thus, $G \times^{H_{\text{red}}} N$ is a monoid scheme; moreover, the natural morphism $G \times^{H_{\text{red}}} N \rightarrow M$ is finite, and bijective on closed points. As $G \times^{H_{\text{red}}} N_{\text{red}} = (G \times^{H_{\text{red}}} N)_{\text{red}}$, this yields our assertions.

(iii) Since M is unit dense with unit group G and ψ is a homeomorphism, we see that $G \times^{H_{\text{red}}} N_{\text{red}}$ is unit dense with unit group G as well. It follows that $G \times N_{\text{red}}$ is unit dense with unit group $G \times H_{\text{red}}$. Thus, H_{red} is the unit group of N_{red} and is dense there.

(iv) Just note that ψ restricts to the natural isomorphism $G \times^{H_{\text{red}}} H_{\text{red}} \xrightarrow{\cong} G$; moreover, $G \times^{H_{\text{red}}} H_{\text{red}}$ is a dense open subset of $G \times^{H_{\text{red}}} N_{\text{red}}$. \square

Example 3.2.6. Assume that $\text{char}(k) = p > 0$. Let

$$M := \{(x, y, z) \in \mathbb{A}^3 \mid z^p = xy^p \text{ and } x \neq 0\}$$

equipped with pointwise multiplication, as in Example 3.2.4. Recall from this example that $G \cong \mathbb{G}_m^2$ and that the universal homomorphism $\varphi : M \rightarrow G/H$ is just $x : M \rightarrow \mathbb{G}_m$, with scheme-theoretic fiber N at 1 being the submonoid scheme $(z^p = y^p)$ of (\mathbb{A}^2, \times) . It follows that $N_{\text{red}} \cong (\mathbb{A}^1, \times)$, $H_{\text{red}} \cong \mathbb{G}_m$ and $G \times^{H_{\text{red}}} N_{\text{red}} \cong \mathbb{G}_m \times \mathbb{A}^1$, where the right-hand side is equipped with pointwise multiplication. Moreover, the map $\psi : G \times^{H_{\text{red}}} N_{\text{red}} \rightarrow M$ is identified with $(t, u) \mapsto (t^p, t, u)$.

Returning to the general setting, we now relate the idempotents and kernel of M with those of N_{red} :

Proposition 3.2.7. *Keep the notation and assumptions of Proposition 3.2.5.*

- (i) $E(M) = E(N_{\text{red}})$.
- (ii) The assignment $I \mapsto I \cap N_{\text{red}}$ defines a bijection between the two-sided ideals of M and those two-sided ideals of N_{red} that are stable under conjugation by G . The inverse bijection is given by $J \mapsto GJ$.
- (iii) We have $\ker(M) \cap N_{\text{red}} = \ker(N_{\text{red}})$ and $\ker(M) = G \ker(N_{\text{red}})$.

Proof. (i) Clearly, $E(N_{\text{red}}) \subseteq E(M)$. Moreover, if $e \in E(M)$ then $\varphi(e) = 1_{G/H}$ and hence $e \in N$, i.e., $e \in N_{\text{red}}$.

(ii) Consider a two-sided ideal I of M . Then $J := I \cap N_{\text{red}}$ is a two-sided ideal of N_{red} , stable under conjugation by G (since so are I and N_{red}). Moreover, $I = GJ$ since $M = GN_{\text{red}}$ and $I = GI$.

Conversely, let J be a two-sided ideal of N_{red} , stable under conjugation by G . Then $I := GJ$ is closed in M and satisfies $I \cap N_{\text{red}} = J$, as follows easily from the fact that the map $G \times^{H_{\text{red}}} N_{\text{red}} \rightarrow M$ is finite and bijective. Moreover, $GIG = GJG = GJ = I$ by stability of J under conjugation. Since M is unit dense and I is closed in M , it follows that $MIM = I$, i.e., I is a two-sided ideal.

(iii) Since N_{red} is stable under conjugation by G , so is $\ker(N_{\text{red}})$. In view of (ii), it follows that $\ker(M) \cap N_{\text{red}} = \ker(N_{\text{red}})$. Together with Proposition 2.4.4, this yields $\ker(M) = G \ker(N_{\text{red}}) G = G \ker(N_{\text{red}})$. \square

Remarks 3.2.8. (i) If N is reduced, then any homomorphism of algebraic monoids from N to an algebraic group is trivial.

Indeed, let $\kappa : N \rightarrow \mathcal{G}$ be such a homomorphism. We may assume that κ is the universal morphism $N \rightarrow H/K$ of Proposition 3.2.1, where K is a normal subgroup scheme of H . Then the G -action on N by conjugation yields a G -action on H/K , compatible with the conjugation action on H ; thus, K is a normal subgroup scheme of G . Moreover, the H -equivariant morphism κ induces a G -equivariant morphism

$$\psi : M = G \times^H N \longrightarrow G \times^H H/K \cong G/K, \quad (g, y)H \longmapsto (g, \kappa(y))H.$$

Also, $\psi(1)$ is the neutral element of G/K , since $\kappa(1)$ is the neutral element of H/K . It follows that $\psi(xy) = \psi(x)\psi(y)$ for all $x \in G$ and $y \in M$, and hence for all $x, y \in M$. By Proposition 3.2.1, ψ factors through φ , and hence $K = H$.

We do not know if the analogous statement holds for N_{red} when N is nonreduced.

(ii) Let M be a unit dense monoid, and M° its neutral component. Then M° is stable under the action of the unit group G on M by conjugation, and $M = GM^\circ$ by Proposition 2.4.3. Thus, $G \times^{G^\circ} M^\circ$ is an algebraic monoid, the disjoint union of the irreducible components of M . Moreover, the map

$$\varphi : G \times^{G^\circ} M^\circ \longrightarrow M, \quad (g, x)G^\circ \longmapsto gx$$

is a homomorphism of algebraic monoids, which is readily seen to be finite and birational. Hence φ is an isomorphism whenever M is normal.

3.3 Structure of irreducible algebraic monoids

We begin this subsection by presenting some classical results on the structure of an arbitrary connected algebraic group G . By a theorem of Chevalley, G has a largest connected affine normal subgroup G_{aff} ; moreover, the quotient group G/G_{aff} is an abelian variety. In other words, G sits in a unique exact sequence of connected algebraic groups

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \xrightarrow{\alpha} A \longrightarrow 1,$$

where G_{aff} is linear, and $A := G/G_{\text{aff}}$ is an abelian variety. This exact sequence is generally nonsplit; yet G has a smallest closed subgroup H such that $\alpha|_H$ is surjective. Moreover, H

is connected, contained in the center of G , and satisfies $\mathcal{O}(H) = k$. In fact, H is the largest closed subgroup of G satisfying the latter property, which defines the class of *anti-affine* algebraic groups; we denote H by G_{ant} . Finally, we have the *Rosenlicht decomposition*: $G = G_{\text{aff}} G_{\text{ant}}$, and $(G_{\text{ant}})_{\text{aff}}$ is the connected neutral component of the scheme-theoretic intersection $G_{\text{aff}} \cap G_{\text{ant}}$. In other words, we have an isomorphism of algebraic groups

$$G \cong (G_{\text{aff}} \times G_{\text{ant}}) / (G_{\text{ant}})_{\text{aff}},$$

and the quotient group scheme $(G_{\text{aff}} \cap G_{\text{ant}}) / (G_{\text{ant}})_{\text{aff}}$ is finite. We refer to [BSU12] for an exposition of these results and of further developments.

We shall obtain a similar structure result for an arbitrary irreducible algebraic monoid; then the unit group is a connected algebraic group by Theorem 2.2.4. Our starting point is the following:

Proposition 3.3.1. *Let M be an irreducible algebraic monoid, G its unit group, $\varphi : M \rightarrow \mathcal{G}(M) = G/H$ the universal homomorphism to an algebraic group, and N the scheme-theoretic fiber of φ at 1. Then H and N are affine.*

Proof. Recall from Proposition 3.2.1 that H is the normal subgroup scheme of G generated by G_x , where $x \in \ker(M)$. Since G_x is the isotropy subgroup scheme of a point for a faithful action of G (the action on M by left multiplication), it follows that G_x is affine (see e.g. [BSU12, Cor. 2.1.9]). The image of G_x under the homomorphism $\alpha : G \rightarrow A$ is affine (as the image of an affine group scheme by a homomorphism of group schemes) and proper (as a subgroup scheme of the abelian variety G/G_{aff}), hence finite. But $\alpha(G_x) = \alpha(H)$ by the definition of H and the commutativity of A ; hence $\alpha(H)$ is finite. Also, the kernel of the homomorphism $\alpha|_H$ is a subgroup scheme of G_{aff} , and hence is affine. Thus, H is an extension of a finite group scheme by an affine group scheme, and hence is affine (see e.g. [BSU12, Lem. 2.1.1, Lem. 2.1.3]).

As a consequence, H_{red} is affine, and hence so is N_{red} by Theorem 3.1.1 and Proposition 3.2.5. It follows that N is affine, in view of [Ha77, Exer. III.3.1]). \square

Remark 3.3.2. If $\text{char}(k) = 0$, then N is reduced, and any homomorphism from N to an algebraic group is trivial by Remark 3.2.8 (i). If in addition N is irreducible (e.g., if M is normal), then $\ker(N)$ is generated by the minimal idempotents. Indeed, the unit group H of N is generated by the conjugates of the isotropy group H_e for some idempotent $e \in \ker(N)$, by Proposition 3.2.1 (ii). So the assertion follows from [Pu06, Thm. 2.1].

A remarkable consequence of Proposition 3.3.1 is the following:

Corollary 3.3.3. *Any irreducible algebraic monoid is quasi-projective.*

Proof. With the notation of the above theorem, the morphism $\varphi : M = G \times^H N \rightarrow G/H$ is affine since so is N . Moreover, G/H is quasiprojective since so is any algebraic group (see e.g. [BSU12, Prop. 3.1.1]). Thus, M is quasiprojective as well. \square

Another consequence is a version of Chevalley's structure theorem for an irreducible algebraic monoid; it generalizes [BR07, Thm. 1.1], where the monoid is assumed to be normal.

Theorem 3.3.4. *Let M be an irreducible algebraic monoid, G its unit group, and M_{aff} the closure of G_{aff} in M .*

- (i) *M_{aff} is an irreducible affine algebraic monoid with unit group G_{aff} .*
- (ii) *The action of G_{aff} on M_{aff} extends to an action of $G = G_{\text{aff}}G_{\text{ant}}$, where G_{ant} acts trivially.*
- (iii) *The natural map $G_{\text{ant}} \times^{G_{\text{ant}} \cap G_{\text{aff}}} M_{\text{aff}} \rightarrow G \times^{G_{\text{aff}}} M_{\text{aff}}$ is an isomorphism of irreducible algebraic monoids. Moreover, the natural map*

$$\kappa : G \times^{G_{\text{aff}}} M_{\text{aff}} \rightarrow M$$

is a finite birational homomorphism of algebraic monoids.

- (iv) *$E(M) = E(M_{\text{aff}})$ and $\ker(M) = G \ker(M_{\text{aff}})$.*
- (v) *M is normal if and only if M_{aff} is normal and κ is an isomorphism. Then the assignment $I \mapsto I \cap M_{\text{aff}}$ defines a bijection between the two-sided ideals of M and those of M_{aff} ; the inverse bijection is given by $J \mapsto GJ$. In particular, $\ker(M) \cap M_{\text{aff}} = \ker(M_{\text{aff}})$.*

Proof. (i) Clearly, M_{aff} is an irreducible submonoid of M , and $G(M_{\text{aff}})$ contains G_{aff} as an open subgroup. Since $G(M_{\text{aff}})$ is connected, it follows that $G(M_{\text{aff}}) = G_{\text{aff}}$. Hence M_{aff} is affine by Theorem 3.1.1.

(ii) follows readily from the Rosenlicht decomposition: since $G_{\text{aff}} \cap G_{\text{ant}}$ is contained in the center of G_{aff} , its action on M_{aff} by conjugation is trivial. Thus, the G_{aff} -action by conjugation on M_{aff} extends to an action of $G \cong (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{aff}} \cap G_{\text{ant}})$, where G_{ant} acts trivially.

(iii) The first assertion follows from the Rosenlicht decomposition again, since that decomposition yields an isomorphism $G \cong G_{\text{ant}} \times^{G_{\text{ant}} \cap G_{\text{aff}}} G_{\text{aff}}$ of principal G_{aff} -bundles over $G/G_{\text{aff}} \cong G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}})$. For the second assertion, note that κ restricts to the natural isomorphism $G \times^{G_{\text{aff}}} G_{\text{aff}} \rightarrow G$, and hence is birational. To show that κ is finite, write $M = G \times^H N$ as in Subsection 3.2. Then H and N are affine by Proposition 3.3.1; also, the natural map $G \times^{H_{\text{red}}} N_{\text{red}} \rightarrow M$ is finite and bijective by Proposition 3.2.5. It follows that the analogous map

$$\gamma : G \times^{H_{\text{red}}^o} N_{\text{red}}^o \longrightarrow M$$

is finite and surjective. But $H_{\text{red}}^o \subseteq G_{\text{aff}}$ since H is affine. Thus,

$$G \times^{H_{\text{red}}^o} N_{\text{red}}^o \cong G \times^{G_{\text{aff}}} (G_{\text{aff}} \times^{H_{\text{red}}^o} N_{\text{red}}^o).$$

Moreover, $N_{\text{red}}^o \subseteq M_{\text{aff}}$ since N_{red}^o is the closure in M of $H_{\text{red}}^o \subseteq G_{\text{aff}}$; hence $G_{\text{aff}} N_{\text{red}}^o \subseteq M_{\text{aff}}$. So γ factors as the natural map

$$\beta : G \times^{G_{\text{aff}}} (G_{\text{aff}} \times^{H_{\text{red}}^o} N_{\text{red}}^o) \rightarrow G \times^{G_{\text{aff}}} M_{\text{aff}}$$

(induced from the map $\delta : G_{\text{aff}} \times^{H_{\text{red}}^o} N_{\text{red}}^o \rightarrow M_{\text{aff}}$), followed by κ . Now δ is the restriction of κ to a closed subvariety, and hence is finite; thus, its image $G_{\text{aff}} N_{\text{red}}^o$ is closed in M_{aff} . But $M_{\text{aff}} = \overline{G_{\text{aff}}}$, and hence δ is surjective. Hence β is finite and surjective. Since $\gamma = \kappa \circ \beta$, it follows that κ is finite and surjective as well.

(iv) Let $e \in E(M)$. By Corollary 2.1.7, e lifts to some idempotent f of $G \times^{G_{\text{aff}}} M_{\text{aff}}$. Then the image of f in G/G_{aff} is the neutral element, and hence $f \in M_{\text{aff}}$ so that $e \in E(M_{\text{aff}})$. The converse is obvious.

Next, choose e minimal. Then $\ker(M) = GeG$ by Proposition 2.4.4, and hence $\ker(M) = G(G_{\text{aff}}eG_{\text{aff}})$ in view of the Rosenlicht decomposition. But $G_{\text{aff}}eG_{\text{aff}} = \ker(M_{\text{aff}})$ since e is a minimal idempotent of G_{aff} .

(v) Assume that M is normal. By (iii) and Zariski's Main Theorem, it follows that κ is an isomorphism. In particular, $G \times^{G_{\text{aff}}} M_{\text{aff}}$ is normal. Since the natural morphism $G \times M_{\text{aff}} \rightarrow G \times^{G_{\text{aff}}} M_{\text{aff}}$ is smooth, it follows that $G \times M_{\text{aff}}$ is normal (e.g., by Serre's criterion); hence so is M_{aff} . The converse is straightforward. This proves the first assertion.

The second assertion is proved by the argument of Proposition 3.2.7 (ii); note that any two-sided ideal of M_{aff} is stable under conjugation by G , in view of (ii) above. \square

Example 3.3.5. Let n be a positive integer, μ_n the group scheme of n th roots of unity, and A an abelian variety containing μ_n as a subgroup scheme (any ordinary elliptic curve will do). As in Example 3.2.4, let N be the submonoid scheme $(z^n = y^n)$ of (\mathbb{A}^2, \times) , and H the unit subgroup scheme of N ; then $H \cong \mu_n \times \mathbb{G}_m$. Next, let $G := A \times \mathbb{G}_m$; this is a commutative connected algebraic group containing H as a subgroup scheme. Finally, let

$$M := G \times^H N = A \times^{\mu_n} N.$$

Then one checks that M is an irreducible algebraic monoid with unit group G . Clearly, $G_{\text{aff}} = \mathbb{G}_m$ and $A(G) = A$; also, one checks that $M_{\text{aff}} = (\mathbb{A}^1, \times)$ and hence $G \times^{G_{\text{aff}}} M_{\text{aff}} \cong A \times \mathbb{A}^1$. The morphism $\kappa : G \times^{G_{\text{aff}}} M_{\text{aff}} \rightarrow M$ sends the closed subscheme $\mu_n \times \{0\}$ to 0, and restricts to an isomorphism over the complement.

In view of Theorem 3.3.4, we may transfer information from affine algebraic monoids (about which much is known, see [Pu88, Re05]) to general ones. For example, the minimal idempotents of any irreducible algebraic monoid are all conjugate under the unit group, since this holds in the affine case by [Pu88, Prop. 6.1, Cor. 6.8]. Another remarkable corollary is the following relation between the partial order on idempotents and limits of one-parameter subgroups, in any irreducible algebraic semigroup:

Corollary 3.3.6. *Let (S, μ) be an irreducible algebraic semigroup, and $e, f \in E(S)$. Then $e \leq f$ if and only if there exists a homomorphism of algebraic semigroups $\lambda : (\mathbb{A}^1, \times) \rightarrow (S, \mu)$ such that $\lambda(0) = e$ and $\lambda(1) = f$.*

Proof. The “if” implication is obvious (and holds in every algebraic semigroup). For the converse, assume that $e \leq f$. Then $e \in fSf$ and the latter is an irreducible algebraic monoid. Thus, we may assume that S itself is an irreducible algebraic monoid, and f is the neutral element. In view of Theorem 3.3.4, we may further assume that S is affine. Then the assertion follows from [Pu81, Thm. 2.9, Thm. 2.10]. \square

3.4 The Albanese morphism

By [Se58, Sec. 4], every irreducible variety X admits a universal morphism to an abelian variety: the *Albanese morphism*,

$$\alpha : X \longrightarrow A(X).$$

The group $A(X)$ is generated by the differences $\alpha(x) - \alpha(y)$, where $x, y \in X$. Also, X admits a universal rational map to an abelian variety: the *Albanese rational map*,

$$\alpha_{\text{rat}} : X \dashrightarrow A(X)_{\text{rat}}.$$

The map α is uniquely determined up to translations and isomorphisms of the algebraic group $A(X)$, and likewise for α_{rat} . Moreover, there exists a unique morphism

$$\beta : A(X)_{\text{rat}} \longrightarrow A(X)$$

such that $\alpha = \beta \circ \alpha_{\text{rat}}$. The morphism β is always surjective; when X is nonsingular, it is an isomorphism. For an arbitrary X , we have $A(X)_{\text{rat}} = A(U)$, where $U \subseteq X$ denotes the nonsingular locus; in particular, α_{rat} is defined at any nonsingular point of X .

When X is equipped with a base point x , we may assume that $\alpha(x)$ is the origin of $A(X)$. If X is nonsingular at x , then we may further assume that $\alpha_{\text{rat}}(x)$ is the origin of $A(X)_{\text{rat}}$. Then α and α_{rat} are unique up to isomorphisms of algebraic groups.

Next, observe that the Albanese morphism of a connected linear algebraic group G is constant: indeed, G is generated by rational curves, and any morphism from such a curve to an abelian variety is constant. For a connected algebraic group G (not necessarily linear), it follows that $\alpha = \alpha_{\text{rat}}$ is the quotient homomorphism by the largest connected affine subgroup G_{aff} . This determines the Albanese rational map of an irreducible algebraic monoid M , which is just the Albanese morphism of its unit group. Some properties of the Albanese morphism of M are gathered in the following:

Proposition 3.4.1. *Let M be an irreducible algebraic monoid with unit group G . Then the map $\alpha : M \rightarrow A(M)$ is a homomorphism of algebraic monoids, and an affine morphism. Moreover, the map $\beta : A(M)_{\text{rat}} = A(G) \rightarrow A(M)$ is an isogeny. If M is normal, then β is an isomorphism.*

Proof. The monoid law $\mu : M \times M \rightarrow M$, $(1_M, 1_M) \mapsto 1_M$ induces a morphism of varieties $A(\mu) : A(M \times M) \rightarrow A(M)$, $0 \mapsto 0$. Since $A(M \times M) \cong A(M) \times A(M)$, it follows that $A(\mu)$ is a homomorphism; hence so is α . In particular, α factors through the universal homomorphism $\varphi : M \rightarrow G/H$ of Proposition 3.2.1. Hence $A(M) = A(G/H) = G/G_{\text{aff}}H$, where $G_{\text{aff}}H$ is a normal subgroup scheme of G such that the quotient $G_{\text{aff}}H/G_{\text{aff}} \cong H/(H \cap G_{\text{aff}})$ is finite. Write $M = G \times^H N$ as in Proposition 3.3.1; then

$$M \cong G \times^{G_{\text{aff}}H} (G_{\text{aff}}H \times^H N)$$

and this identifies α with the natural map to $G/G_{\text{aff}}H$, with fiber $G_{\text{aff}}H \times^H N$. But that fiber is affine, since so are N and $G_{\text{aff}}H/H \cong G_{\text{aff}}/(G_{\text{aff}} \cap H)$. It follows that the morphism α is affine. Also, β is identified with the natural homomorphism $G/G_{\text{aff}} \rightarrow G/G_{\text{aff}}H$; hence the kernel of β is isomorphic to $G_{\text{aff}}H/H$, a finite group scheme.

If M is normal, then $M \cong G \times^{G_{\text{aff}}} M_{\text{aff}}$ by Theorem 3.3.4. Thus, the natural map $M \rightarrow G/G_{\text{aff}}$ is the Albanese morphism. \square

Consider for instance the monoid M constructed in Example 3.3.5. Then $A(M) \cong A/\mu_n$ and $A(G) \cong A$; this identifies β to the quotient morphism $A \rightarrow A/\mu_n$.

Returning to our general setting, recall that every irreducible algebraic monoid may be viewed as an equivariant embedding of its unit group. For an arbitrary equivariant embedding X of a connected algebraic group G , we may again identify $A(X)_{\text{rat}}$ with $A(G)$; when X is normal, we still have $A(X) = A(X)_{\text{rat}}$ as a consequence of [Br10, Thm. 3]. But the morphism α is generally nonaffine, and the finiteness of β is an open question in this setting.

We now characterize algebraic monoids among equivariant embeddings:

Theorem 3.4.2. *Let X be an equivariant embedding of a connected algebraic group G . Then X has a structure of algebraic monoid with unit group G if and only if the Albanese morphism $\alpha : X \rightarrow A(X)$ is affine.*

Proof. In view of Proposition 3.4.1, it suffices to show that X is an algebraic monoid if α is affine. Note that α is $G \times G$ -equivariant for the given action of $G \times G$ on X , and a transitive action on $A(X)$. It follows that $A(X) \cong (G \times G)/(K \times K)\text{diag}(G)$ for a unique normal subgroup scheme K of G ; then $A(X) \cong G/K$ equivariantly for the left (or right) action of G . Moreover, α is a fiber bundle of the form

$$G \times G \times^{(K \times K)\text{diag}(G)} Y \rightarrow (G \times G)/(K \times K)\text{diag}(G),$$

where Y is a scheme equipped with an action of $(K \times K)\text{diag}(G)$; for the left (or right) G -action, this yields the fiber bundle $G \times^K Y \rightarrow G/K$. Also, Y meets the open orbit $G \cong (G \times G)/\text{diag}(G)$ along a dense open subscheme isomorphic to K , where $K \times K$ acts by left and right multiplication, and $\text{diag}(G)$ by conjugation. Thus, the group scheme K is quasi-affine, and hence is affine.

We now show that the group law $\mu_K : K \times K \rightarrow K$ extends to a morphism $\mu_Y : Y \times Y \rightarrow Y$, by following the argument of [Ri98, Prop. 1]. The left action $K \times Y \rightarrow Y$ and the right action $Y \times K \rightarrow Y$ restrict both to μ_K , and hence yield a morphism $(K \times Y) \cup (Y \times K) \rightarrow Y$. Since Y is affine, it suffices to show the equality

$$\mathcal{O}((K \times Y) \cup (Y \times K)) = \mathcal{O}(Y \times Y).$$

But $\mathcal{O}(Y \times Y) = \mathcal{O}(Y) \otimes \mathcal{O}(Y) \subseteq \mathcal{O}(K) \otimes \mathcal{O}(K) = \mathcal{O}(K \times K)$, since K is dense in Y . Moreover,

$$\mathcal{O}((K \times Y) \cup (Y \times K)) = (\mathcal{O}(K) \otimes \mathcal{O}(Y)) \cap (\mathcal{O}(Y) \otimes \mathcal{O}(K)),$$

where the intersection is considered in $\mathcal{O}(K) \otimes \mathcal{O}(K)$. Now for any vector space V and subspace W , we easily obtain the equality $(W \otimes V) \cap (V \otimes W) = W \otimes W$ in $V \otimes V$. When applied to $\mathcal{O}(Y) \subseteq \mathcal{O}(K)$, this yields the desired equality.

Since μ_Y is associative on the dense subscheme K , it is associative everywhere; likewise, it admits 1_K as a neutral element. Thus, μ_Y is an algebraic monoid law on Y . We may now form the induced monoid $G \times^K Y$ as in Subsection 3.2, to get the desired structure on X . \square

3.5 Algebraic semigroups and monoids over perfect fields

In this subsection, we extend most of the above results to the setting of algebraic semigroups and monoids defined over a perfect field. We use the terminology and results of [Sp98], especially Chap. 11 which discusses basic rationality results on varieties.

Let F be a subfield of the algebraically closed field k . We assume that F is *perfect*, i.e., every algebraic extension of F is separable; we denote by \bar{F} the algebraic closure of F in k , and by Γ the Galois group of \bar{F} over F .

We say that an algebraic semigroup (S, μ) (over k) is *defined over F* , or an *algebraic F -semigroup*, if S is an F -variety and the morphism μ is defined over F . Then the set of \bar{F} -points $S(\bar{F})$ is a semigroup equipped with an action of Γ by semigroup automorphisms, and the fixed point subset $S(\bar{F})^\Gamma$ is the semigroup of F -points, $S(F)$.

Note that an algebraic F -semigroup may well have no F -point; for example, an F -variety without F -point equipped with the trivial semigroup law μ_l or μ_r . But this is the only obstruction to the existence of F -idempotents, as shown by the following:

Proposition 3.5.1. *Let (S, μ) be an algebraic F -semigroup.*

- (i) *$E(S)$ and $\ker(S)$ (viewed as closed subsets of S) are defined over F .*
- (ii) *If S is commutative, then its smallest idempotent is defined over F .*
- (iii) *If S has an F -point, then it has an idempotent F -point.*

Proof. (i) Clearly, $E(S)$ and $\ker(S)$ are defined over \bar{F} and their sets of \bar{F} -points are stable under the action of Γ on $S(\bar{F})$. Thus, $E(S)$ and $\ker(S)$ are defined over F by [Sp98, Prop. 11.2.8(i)].

(ii) is proved similarly.

(iii) Let $x \in S(F)$ and denote by $\langle x \rangle$ the closure in S of the set $\{x^n, n \geq 1\}$. Then $\langle x \rangle$ is a closed commutative subsemigroup of S , defined over F by [Sp98, Lem. 11.2.4]. In view of (ii), $\langle x \rangle$ contains an idempotent defined over F . \square

We do not know if any algebraic F -semigroup S has a minimal idempotent defined over F . This holds if S is irreducible, as we will see in Proposition 3.5.3. First, we record two rationality results on algebraic monoids:

Proposition 3.5.2. *Let $(M, \mu, 1_M)$ be an algebraic monoid with unit group G and neutral component M° . If M and μ are defined over F , then so are 1_M , G and M° . Moreover, the inverse map $\iota : G \rightarrow G$ is defined over F .*

Proof. Observe that 1_M is the unique point $x \in M$ such that $xy = yx = y$ for all $y \in M(\bar{F})$ (since $M(\bar{F})$ is dense in M). It follows that $1_M \in M(\bar{F})$; also, M is Γ -invariant by uniqueness. Thus, $1_M \in M(F)$.

The assertion on G follows from [Sp98, Prop. 11.2.8(ii)]. It implies that G° is defined over F by [loc. cit., Prop. 12.1.1]. Since M° is the closure of G° in M , it is also defined over F in view of [loc. cit., Prop. 11.2.8(i)].

It remains to show that ι is defined over F ; equivalently, its graph is an F -subvariety of $G \times G$. But this graph equals

$$\{(x, y) \in G \times G \mid xy = 1_M\} = \mu_G^{-1}(1_M),$$

where $\mu_G : G \times G \rightarrow G$ denotes the restriction of μ , and $\mu_G^{-1}(1_M)$ stands for the set-theoretic fiber. Moreover, this fiber is defined over F in view of [Sp98, Cor. 11.2.14]. \square

Proposition 3.5.3. *Let $(M, \mu, 1_M)$ be an irreducible algebraic monoid with unit group G . If (M, μ) is defined over F , then the universal homomorphism to an algebraic group, $\varphi : M \rightarrow \mathcal{G}(M)$, is defined over F as well. Moreover, G_{aff} and M_{aff} are defined over F .*

Proof. The first assertion follows from the uniqueness of φ by a standard argument of Galois descent, see [Se59, Chap. V, §4]. The (well-known) assertion on G_{aff} is proved similarly; it implies the assertion on M_{aff} by [Sp98, Prop. 11.2.8(i)]. \square

Returning to algebraic semigroups, we obtain the promised:

Proposition 3.5.4. *Let (S, μ) be an irreducible algebraic F -semigroup. If S has an F -point, then some minimal idempotent of S is defined over F .*

Proof. By Proposition 3.5.1, we may choose $e \in E(S(F))$. Then eSe is a closed irreducible submonoid of S , and is defined over S in view of [Sp98, Prop. 11.2.8(i)] again. Moreover, any minimal idempotent of eSe is a minimal idempotent of S . So we may assume that S is an irreducible monoid, M . In view of Theorem 3.3.4 and Proposition 3.5.3, we may further assume that M is affine. Then the unit group of M contains a maximal torus T defined over F , by Proposition 3.5.2 and [Sp98, Thm. 13.3.6, Rem. 13.3.7]. The closure \bar{T} of T in M is defined over F , and meets $\ker(M)$ in view of [Pu88, Cor. 6.10]. So the (set-theoretic) intersection $N := \bar{T} \cap \ker(M)$ is a commutative algebraic semigroup, defined over F by [Sp98, Thm. 11.2.13]. Now applying Proposition 3.5.1 to N yields the desired idempotent. \square

Remark 3.5.5. The above observations leave open all the rationality questions for an algebraic semigroup S over a field F , not necessarily perfect. In fact, S has an idempotent F -point if it has an F -point, as follows from the main result of [BR12]. But some results do not extend to this setting: for example, the kernel of an algebraic F -monoid may not be defined over F , as shown by a variant of the standard example of a linear algebraic F -group whose unipotent radical is not defined over F (see [SGA3, Exp. XVII, 6.4.a]) or [Sp98, 12.1.6]; specifically, replace the multiplicative group \mathbb{G}_m with the monoid (\mathbb{A}^1, \times) in the construction of this example). Also, note that Chevalley's structure theorem fails over any imperfect field (see [SGA3, Exp. XVII, App. III, Cor.], and [To11] for recent developments). Thus, G_{aff} may not be defined over F with the notation and assumptions of Proposition 3.5.3. Yet the Albanese morphism still exists for any F -variety equipped with an F -point (see [Wi08, App. A]) and hence for any algebraic F -semigroup equipped with an F -idempotent.

4 Algebraic semigroup structures on certain varieties

4.1 Abelian varieties

In this subsection, we begin by describing all the algebraic semigroup laws on an abelian variety. Then we apply the result to the study of the Albanese morphism of an irreducible algebraic semigroup.

Proposition 4.1.1. *Let $(A, +)$ be an abelian variety, μ an algebraic semigroup law on A , and e an idempotent of (A, μ) ; choose e as the neutral element of $(A, +)$.*

(i) *There exists a unique decomposition*

$$(A, \mu) = (A_0, \mu_0) \times (A_l, \mu_l) \times (A_r, \mu_r) \times (B, +)$$

where A_0, A_l, A_r and B are abelian varieties, and μ_0 (resp. μ_l, μ_r) the trivial semigroup law on A_0 (resp. A_l, A_r) defined in Example 2.1.8 (i).

(ii) *The corresponding projection $\varphi : A \rightarrow B$ is the universal homomorphism of (A, μ) to an algebraic group. Moreover, we have $E(S) = \{e\} \times A_l \times A_r \times \{e\}$ and $\ker(S) = \{e\} \times A_l \times A_r \times B$.*

Proof. (i) By [Mu74, Chap. II, §4, Cor. 1], the morphism $\mu : A \times A \rightarrow A$ satisfies

$$\mu(x, y) = \varphi(x) + \psi(y) + x_0,$$

where $x_0 \in A$ and φ, ψ are endomorphisms of the algebraic group A . Since $\mu(e, e) = e$ and $\varphi(e) = \psi(e) = e$, we have $x_0 = e$, i.e., $\mu(x, y) = \varphi(x) + \psi(y)$. Now the associativity of μ is equivalent to the equality

$$\varphi \circ \varphi(x) + \varphi \circ \psi(y) + \psi(z) = \varphi(x) + \psi \circ \varphi(y) + \psi \circ \psi(z),$$

that is, to the equalities

$$\varphi \circ \varphi = \varphi, \quad \varphi \circ \psi = \psi \circ \varphi, \quad \psi \circ \psi = \psi.$$

This easily yields the desired decomposition, where $A_0 := \text{Ker}(\varphi) \cap \text{Ker}(\psi)$, $A_l := \text{Im}(\varphi) \cap \text{Ker}(\psi)$, $A_r := \text{Ker}(\varphi) \cap \text{Im}(\psi)$, and $B := \text{Im}(\varphi) \cap \text{Im}(\psi)$, so that φ (resp. ψ) is the projection of A to $A_l \times B$ (resp. $A_r \times B$). The uniqueness of this decomposition follows from that of φ and ψ .

(ii) Let $\gamma : (A, \mu) \rightarrow \mathcal{G}$ be a homomorphism to an algebraic group. Then the image of γ is a complete irreducible variety, and hence generates an abelian subvariety of \mathcal{G} . Thus, we may assume that \mathcal{G} is an abelian variety, say A' with group law also denoted additively. As above, we have $\gamma(x) = \pi(x) + x'_0$, where $\pi : A \rightarrow A'$ is a homomorphism of algebraic groups and $x'_0 \in A'$. Since $\gamma(e)$ is idempotent, we obtain $x'_0 = 0$, i.e., $\gamma : (A, +) \rightarrow A'$ is also a homomorphism. It follows readily that γ sends $A_0 \times A_l \times A_r \times \{e\}$ to 0. So γ factors as φ followed by a unique homomorphism $\gamma' : B \rightarrow A'$. This proves the assertion on φ ; those on $E(S)$ and $\ker(S)$ are easily checked. \square

Proposition 4.1.2. *Let (S, μ) be an irreducible algebraic semigroup, $e \in E(S)$, and $\alpha : S \rightarrow A(S)$ the Albanese morphism; we assume that $\alpha(e) = 0$.*

(i) *There exists a unique algebraic semigroup law $A(\mu)$ on $A(S)$ such that α is a homomorphism.*

(ii) *Let $\varphi : (A(S), A(\mu)) \rightarrow B(S)$ be the universal homomorphism to an algebraic group. Then the map $\varphi \circ \alpha : eSe \rightarrow B(S)$ is the Albanese morphism.*

Proof. (i) follows from the functorial properties of the Albanese morphism (see [Se58, Sec. 2]) by arguing as in the beginning of the proof of Proposition 3.4.1.

(ii) Consider the inclusions $eSe \subseteq eS \subseteq S$. Each of them admits a retraction, $x \mapsto xe$ (resp. $x \mapsto ex$). Thus, the corresponding morphisms $A(eSe) \rightarrow A(eS) \rightarrow A(S)$ also admit retractions, and hence are closed immersions. So we may identify $A(eSe)$ with the subgroup of $A(S)$ generated by the differences $\alpha(exe) - \alpha(eye)$, where $x, y \in S$. But $\alpha(exe) = A(\mu)(\alpha(e), A(\mu)(\alpha(x), \alpha(e)))$ and $\alpha(e)$ is of course an idempotent of $(A(S), A(\mu))$. Hence $\alpha(e) = (e, a_l, a_r, e)$ in the decomposition of Proposition 4.1.1. Using that decomposition, we obtain $\alpha(exe) = (e, a_l, a_r, b(x))$, where $b(x)$ denotes the projection of $\alpha(x)$ to $B(S)$. This yields the desired identification of $A(eSe)$ to $B(S)$. \square

Combined with Proposition 3.4.1, the above result yields:

Corollary 4.1.3. *Let S be an irreducible algebraic semigroup.*

(i) *All the maximal submonoids of S have the same Albanese variety, and all the maximal subgroups have isogenous Albanese varieties.*

(ii) *The irreducible monoid eSe is affine for all $e \in E(S)$ if this holds for some $e \in E(S)$ (e.g., if S has a zero element).*

Remarks 4.1.4. (i) As a special case of the above corollary, every irreducible algebraic monoid having a zero element is affine. This also follows from Theorem 3.1.1, since the unit group is affine in view of [BSU12, Cor. 2.1.9].

(ii) With the notation and assumptions of Proposition 4.1.2, the map $\varphi \circ \alpha : S \rightarrow B(S)$ is the universal homomorphism to an abelian variety. Also, recall from Remark 3.2.2 that there exists a universal homomorphism to an algebraic group, $\psi : S \rightarrow \mathcal{G}(S)$, and that $\mathcal{G}(S)$ is connected. It follows that $B(S)$ is the Albanese variety of $\mathcal{G}(S)$.

(iii) Consider an irreducible algebraic semigroup (S, μ) and its rational Albanese map $\alpha_{\text{rat}} : S \rightarrow A(S)_{\text{rat}}$. If the image of $\mu : S \times S \rightarrow S$ meets the domain of definition of α_{rat} , then there exists a unique algebraic semigroup structure $A(\mu)$ on $A(S)_{\text{rat}}$ such that α_{rat} is a ‘rational homomorphism’, i.e., $\alpha_{\text{rat}}(\mu(x, y)) = A(\mu)(\alpha_{\text{rat}}(x), \alpha_{\text{rat}}(y))$ whenever α_{rat} is defined at $x, y \in S$ and at $\mu(x, y)$ (as can be checked by the argument of Proposition 4.1.2). But this does not hold for an arbitrary S ; for example, if $S \subseteq \mathbb{A}^3$ is the affine cone over an elliptic curve $E \subseteq \mathbb{P}^2$ and if $\mu = \mu_0$. Here 0 , the origin of \mathbb{A}^3 , is the unique singular point of S , and α_{rat} is the natural map $S \setminus \{0\} \rightarrow E$.

4.2 Irreducible curves

In this subsection, we classify the irreducible algebraic semigroups of dimension 1; those having a nontrivial law (as defined in Example 2.1.8 (i)) turn out to be algebraic monoids.

Such semigroups include of course the connected algebraic groups of dimension 1, presented in Example 2.1.9 (iv). We now construct further examples: let (a_1, \dots, a_n) be a strictly increasing sequence of positive integers having no nontrivial common divisor, and consider the map

$$\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^n, \quad x \longmapsto (x^{a_1}, \dots, x^{a_n}).$$

Then φ is a homomorphism of algebraic monoids, where \mathbb{A}^1 and \mathbb{A}^n are equipped with pointwise multiplication. Also, one checks that the morphism φ is finite; hence its image is a closed submonoid of \mathbb{A}^n , containing the origin as its zero element. We denote this monoid by $M(a_1, \dots, a_n)$, and call it an *affine monomial curve*; it only depends on the abstract submonoid of $(\mathbb{Z}, +)$ generated by a_1, \dots, a_n . One may check that φ restricts to an isomorphism $\mathbb{A}^1 \setminus \{0\} \xrightarrow{\cong} M(a_1, \dots, a_n) \setminus \{0\}$; also, $M(a_1, \dots, a_n)$ is singular at the origin unless φ is an isomorphism, i.e., unless $a_1 = 1$.

Theorem 4.2.1. *Let S be an irreducible curve, and μ a nontrivial algebraic semigroup structure on S . Then (S, μ) is either an algebraic group or an affine monomial curve.*

Proof. As the arguments are somewhat long and indirect, we divide them into four steps.

Step 1: we show that every idempotent of S is either a neutral or a zero element.

Let $e \in E(S)$. Since Se is a closed irreducible subvariety of S , it is either the whole S or a single point; in the latter case, $Se = \{e\}$. Thus, one of the following cases occurs:

- (i) $Se = eS = S$. Then any $x \in S$ satisfies $xe = ex = x$, i.e., e is the neutral element.
- (ii) $Se = \{e\}$ and $eS = S$. Then for any $x, y \in S$, we have $xe = e$ and $ey = y$. Thus, $xy = xey = ey = y$. So $\mu = \mu_r$ in the notation of Example 2.1.8 (i), a contradiction since μ is assumed to be nontrivial.
- (iii) $eS = \{e\}$ and $Se = S$. This case is excluded similarly.
- (iv) $Se = eS = \{e\}$. Then e is the zero element of S .

Step 2: we show that if S is complete, then it is an elliptic curve.

For this, we first reduce to the case where S has a zero element. Otherwise, S has a neutral element by Step 1. Hence S is a monoid with unit group G being \mathbb{G}_a , \mathbb{G}_m or an elliptic curve, in view of the classification of connected algebraic groups of dimension 1. In the latter case, G is complete and hence $G = S$. On the other hand, if $G = \mathbb{G}_a$ or \mathbb{G}_m , then $S \setminus G$ is a nonempty closed subsemigroup of S in view of Proposition 2.2.5. Hence $S \setminus G$ contains an idempotent, which must be the zero element of S by Step 1. This yields the desired reduction.

The semigroup law $\mu : S \times S \rightarrow S$ sends $S \times \{0\}$ to the point 0. By the rigidity lemma (see e.g. [Mu74, Chap. II, §4]), it follows that $\mu(x, y) = \varphi(y)$ for some morphism $\varphi : S \rightarrow S$. The associativity of μ yields

$$\varphi(z) = (xy)z = x(yz) = \varphi(yz) = \varphi(\varphi(z))$$

for all $x, y, z \in S$; hence φ is a retraction to its image. Since S is an irreducible curve, either $\varphi = \text{id}$ or the image of φ consists of a single point x . In the former case, $\mu = \mu_r$, whereas $\mu = \mu_x$ in the latter case. Thus, μ is trivial, a contradiction.

Step 3: we show that if S is an affine monoid, then it is isomorphic to \mathbb{G}_a , \mathbb{G}_m or an affine monomial curve.

We may view S as an equivariant embedding of its unit group G , and that group is either \mathbb{G}_a or \mathbb{G}_m . Since $\mathbb{G}_a \cong \mathbb{A}^1$ as a variety, any affine equivariant embedding of \mathbb{G}_a is \mathbb{G}_a itself. So we may assume that $G = \mathbb{G}_m$. Then the coordinate ring $\mathcal{O}(S)$ is a subalgebra

of $\mathcal{O}(\mathbb{G}_m) = k[x, x^{-1}]$, stable under the natural action of \mathbb{G}_m . It follows that $\mathcal{O}(S)$ has a basis consisting of Laurent monomials, and hence that

$$\mathcal{O}(S) = \bigoplus_{n \in \mathcal{M}} x^n,$$

where \mathcal{M} is a submonoid of $(\mathbb{Z}, +)$. Moreover, since \mathbb{G}_m is open in S , the fraction field of $\mathcal{O}(S)$ is the field of rational functions $k(t)$; it follows that \mathcal{M} generates the group \mathbb{Z} . Thus, either $\mathcal{M} = \mathbb{Z}$, or \mathcal{M} is generated by finitely many integers, all of the same sign and having no nontrivial common divisor. In the former case, $S = \mathbb{G}_m$; in the latter case, S is an affine monomial curve.

Step 4: in view of Step 2, we may assume that the irreducible curve S is noncomplete, and hence is affine. Then it suffices to show that S has a nonzero idempotent: then S is an algebraic monoid by Step 1, and we conclude by Step 3. We may further assume that S is nonsingular: indeed, by the nontriviality assumption, the semigroup law $\mu : S \times S \rightarrow S$ is dominant. Using Proposition 2.5.1, it follows that the normalization \tilde{S} (an irreducible nonsingular curve) has a compatible algebraic semigroup structure; then the image of a nonzero idempotent of \tilde{S} is a nonzero idempotent of S .

So we assume that S is an affine irreducible nonsingular semigroup of dimension 1, having a zero element 0, and show that S has a neutral element. Consider the “right regular representation” of S , i.e., its action on the coordinate ring $\mathcal{O}(S)$ by right multiplication: specifically, an arbitrary point $x \in S$ acts on $\mathcal{O}(S)$ by sending a regular function f on S to the regular function $x \cdot f : y \mapsto f(yx)$. This yields a map

$$\varphi : S \longrightarrow \text{End}(\mathcal{O}(S)), \quad x \longmapsto x \cdot$$

which is readily seen to be a homomorphism of abstract semigroups. Moreover, the action of S on $\mathcal{O}(S)$ stabilizes the maximal ideal \mathfrak{m} of 0, and all its powers \mathfrak{m}^n . This defines compatible homomorphisms of abstract semigroups

$$\varphi_n : S \longmapsto \text{End}(\mathcal{O}(S)/\mathfrak{m}^n) \quad (n \geq 1).$$

Since S is a nonsingular curve, we have compatible isomorphisms of k -algebras

$$\mathcal{O}(S)/\mathfrak{m}^n \cong \mathcal{O}_{S,0}/\mathfrak{m}^n \mathcal{O}_{S,0} \cong k[t]/t^n k[t],$$

where $\mathcal{O}_{S,0}$ denotes the local ring of S at 0, and t a generator of the maximal ideal $\mathfrak{m}\mathcal{O}_{S,0}$ of that local ring; the right-hand side is the algebra of truncated polynomials at the order n . This yields compatible isomorphisms of abstract semigroups

$$\text{End}(\mathcal{O}(S)/\mathfrak{m}^n) \xrightarrow{\cong} tk[t]/t^n k[t], \quad \gamma \longmapsto \gamma(t),$$

where the semigroup law on the right-hand side is the composition of truncated polynomials. Thus, we obtain compatible homomorphisms of abstract semigroups

$$\psi_n : S \longrightarrow tk[t]/t^n k[t].$$

In fact, the right-hand side is an algebraic semigroup, and ψ_n is a morphism: indeed, for any $f \in \mathcal{O}(S)$, we have $f(yx) = \sum_{i \in I} f_i(x)g_i(y)$ for some finite collection of functions $f_i, g_i \in \mathcal{O}(S)$ (since the semigroup law is a morphism). Thus, $x \cdot f = \sum_{i \in I} f_i(x)g_i$ and hence the matrix coefficients of the action of x in $\mathcal{O}(S)/\mathfrak{m}^n$ are regular functions of x .

We claim that there exists $n \geq 1$ such that $\psi_n \neq 0$. Otherwise, we have $\varphi_n(x) = 0$ for all $n \geq 1$ and all $x \in S$. Since $\bigcap_n \mathfrak{m}^n = 0$, it follows that $\varphi(x) = \varphi(0)$ for all x , i.e., $f(yx) = f(0)$ for all $f \in \mathcal{O}(S)$ and all $x, y \in S$. Thus, $yx = 0$, i.e., $\mu = \mu_0$; a contradiction.

Now let n be the smallest integer such that $\psi_n \neq 0$. Then ψ_n sends S to the quotient $t^{n-1}k[t]/t^n k[t]$, i.e., to the semigroup of endomorphisms of the algebra $k[t]/t^n k[t]$ given by $t \mapsto ct^{n-1}$, where $c \in k$. If $n \geq 3$, then the composition of any two such endomorphisms is 0, and hence $\psi_n(xy) = 0$ for all $x, y \in S$. Thus, xy belongs to the fiber of ψ_n at 0, a finite set containing 0. Since S is irreducible, it follows that $xy = 0$, i.e., $\mu = \mu_0$; a contradiction. Thus, we must have $n = 2$, and we obtain a nonconstant morphism $\psi = \psi_2 : S \rightarrow \mathbb{A}^1$, where the semigroup law on \mathbb{A}^1 is the multiplication. The image of ψ contains a nonempty open subset U of the unit group \mathbb{G}_m . Then $UU = \mathbb{G}_m$ and hence ψ is surjective. By Proposition 2.1.6, it follows that there exists an idempotent $e \in S$ such that $\psi(e) = 1$. Then e is the desired nonzero idempotent. \square

Remark 4.2.2. One may also deduce the above theorem from the description of algebraic semigroup structures on abelian varieties (Proposition 4.1.1), when the irreducible curve S is assumed to be nonsingular and nonrational. Then the Albanese morphism of S is a locally closed embedding in its Jacobian variety A . It follows that A has no trivial summand A_0 , A_l or A_r (otherwise, the projection to that summand is constant since μ is nontrivial; as the differences of points of S generate the group A , this is a contradiction). In other words, the inclusion of S into A is a homomorphism for a suitable choice of the origin of A . This implies that $S = A$, and we conclude that S is an elliptic curve equipped with its group law.

4.3 Complete irreducible varieties

In this subsection, we obtain a description of all complete irreducible algebraic semigroups, analogous to that of the kernels of algebraic semigroups presented in Proposition 2.3.3:

Theorem 4.3.1. *There is a bijective correspondence between the following:*

- The triples (S, μ, e) , where S is a complete irreducible variety, μ an algebraic semigroup structure on S , and e an idempotent of (S, μ) .
- The tuples $(X, Y, G, \iota, \rho, x_o, y_o)$, where X (resp. Y) is a complete irreducible variety equipped with a point x_o (resp. y_o), G is an abelian variety, $\iota : X \times G \times Y \rightarrow S$ is a closed immersion, and $\rho : S \rightarrow X \times G \times Y$ a retraction of ι .

This correspondence assigns to any such tuple, the algebraic semigroup structure ν on $X \times G \times Y$ defined by

$$\nu((x, g, y), (x', g', y')) := (x, gg', y')$$

and then the algebraic semigroup structure μ on S defined by

$$\mu(s, s') := \iota(\nu(\rho(s), \rho(s'))).$$

The idempotent is $e := \iota(x_o, 1_G, y_o)$. Moreover, ι and ρ are homomorphisms of algebraic semigroups.

The inverse correspondence will be constructed at the end of the proof. We begin that proof with two preliminary results.

Lemma 4.3.2. *Let S be a complete irreducible algebraic semigroup, and $e \in E(S)$. Then $xy = xey$ for all $x, y \in S$.*

Proof. Recall that the map $\varphi : S \rightarrow eS$, $x \mapsto ex$ is a retraction of complete irreducible varieties. We claim that its fibers are connected.

Consider indeed the Stein factorization of φ as the composite

$$S \xrightarrow{\varphi'} X \xrightarrow{\eta} eS,$$

where X denotes the Spec of the sheaf of \mathcal{O}_{eS} -algebras $\varphi_*(\mathcal{O}_S)$ (see [Ha77, Cor. III.11.5]). Then φ' is surjective with connected fibers; in particular, X is an irreducible variety. Moreover, η is finite. Since the inclusion $\iota : eS \rightarrow S$ is a section of φ , the map $\varphi' \circ \iota$ is a section of η . By the irreducibility of X , it follows that η is an isomorphism; this yields our claim.

Next, let F be a (set-theoretic) fiber of φ . Then the morphism $\mu : S \times S \rightarrow S$, $(x, y) \mapsto xy$ sends $\{e\} \times F$ to a point. By the rigidity lemma, $\mu(\{x\} \times F)$ consists of a single point for any $x \in S$. Thus, the map $y \mapsto xy$ is constant on the fibers of φ . Since $\varphi(y) = \varphi(ey)$ for all $y \in S$, we obtain that $xy = xey$. \square

Lemma 4.3.3. *Let S be a complete irreducible algebraic semigroup, and $e \in E(S)$.*

- (i) *The closed submonoid eSe of S is an abelian variety.*
- (ii) *The map $\varphi : S \rightarrow eSe$, $x \mapsto exe$ is a retraction of algebraic semigroups.*
- (iii) *The above map φ is the universal homomorphism to an algebraic group.*

Proof. (i) By Proposition 2.2.5 (iii), it suffices to show that e is the unique idempotent of eSe . But if $f \in E(eSe)$, then $xy = xfy$ for all $x, y \in S$, by Lemma 4.3.2. Taking $x = y = e$ yields $e = efe = f$.

(ii) By Lemma 4.3.2 again, we have $exye = exeye = (exe)(eye)$ for all $x, y \in S$.

(iii) Let \mathcal{G} be an algebraic group and let $\psi : S \rightarrow \mathcal{G}$ be a homomorphism of algebraic semigroups. Then $\psi(e) = 1$ and hence $\psi(x) = \psi(exe)$ for all $x \in S$. Thus, ψ factors uniquely as the homomorphism φ followed by some homomorphism of algebraic groups $eSe \rightarrow \mathcal{G}$. \square

Remark 4.3.4. By the above lemma, every idempotent e of a complete irreducible algebraic semigroup (S, μ) is minimal. Moreover, by Lemma 4.3.2, the image of the morphism μ is exactly the kernel of S ; this is a simple algebraic semigroup in view of Proposition 2.3.3. One may thus deduce part of Theorem 4.3.1 from the structure of simple algebraic semigroups presented in Remark 2.3.5 (i). Yet we will provide a direct, self-contained proof by adapting the arguments of Proposition 2.3.3.

PROOF OF THEOREM 4.3.1.

One readily checks that the map ν (resp. μ) as in the statement yields an algebraic semigroup structure on $X \times G \times Y$ (resp. on S); compare with Example 2.1.8 (ii).

Conversely, given (S, μ, e) as in the statement, consider

$$X := {}_e S e, \quad G := e S e, \quad Y := e S_e$$

with the notation of Remark 2.1.5 (ii). Then G is an abelian variety by Lemma 4.3.3. Let $\iota : X \times G \times Y \rightarrow S$ denote the multiplication map: $\iota(x, g, y) = xgy$. Finally, define a map $\rho : S \rightarrow S \times G \times S$ by

$$\rho(s) = (s(ese)^{-1}, ese, (ese)^{-1}s).$$

Then $s(ese)^{-1} \in X$ since $es(ese)^{-1} = ese(ese)^{-1} = e$ and $s(ese)^{-1}e = s(ese)^{-1}$. Likewise, $(ese)^{-1}s \in Y$. So we may view ρ as a morphism to $X \times G \times Y$.

We claim that $\rho \circ \iota$ is the identity of $X \times G \times Y$. Indeed, $(\rho \circ \iota)(x, g, y) = \rho(xgy)$. Moreover, $exgye = g$ so that

$$\rho(xgy) = (xgyg^{-1}, g, g^{-1}gy).$$

Now $xgyg^{-1} = xgyeg^{-1} = xgeg^{-1} = xe = x$ and likewise, $g^{-1}xgy = y$. This proves the claim.

By that claim, ι is a closed immersion, and ρ a retraction of ι . Also, we have for any $x, x' \in X$, $g, g' \in G$ and $y, y' \in Y$:

$$xgyx'g'y' = xgyex'g'y' = xgex'g'y' = xgeg'y'.$$

In other words, ι is a homomorphism of algebraic semigroups, where $X \times G \times Y$ is given the semigroup structure ν as in the statement.

We next claim that ρ is a homomorphism of algebraic semigroups as well. Indeed,

$$\rho(ss') = (ss'(ess'e)^{-1}, ess'e, (ess'e)^{-1}ss')$$

and hence, using Lemma 4.3.3,

$$\rho(ss') = (ss'(es'e)^{-1}(ese)^{-1}, eses'e, (es'e)^{-1}(ese)^{-1}ss').$$

Moreover,

$$ss'(es'e)^{-1}(ese)^{-1} = ses'e(es'e)^{-1}(ese)^{-1} = se(ese)^{-1} = s(ese)^{-1}$$

by Lemma 4.3.2, and likewise $(es'e)^{-1}(ese)^{-1}ss' = (es'e)^{-1}s'$. Thus,

$$\rho(ss') = (s(ese)^{-1}, eses'e, (es'e)^{-1}s') = \nu(\rho(s), \rho(s'))$$

as required.

Finally, we claim that $ss' = \iota(\nu(\rho(s), \rho(s')))$. Indeed, the right-hand side equals

$$s(ese)^{-1}eses'e(es'e)^{-1}s' = ses' = s$$

in view of Lemma 4.3.2 again.

Remarks 4.3.5. (i) The description of algebraic semigroup laws on a given abelian variety A (Proposition 4.1.1) may of course be deduced from Theorem 4.3.1: with the notation of that theorem, the inclusion ι and retraction ρ yield a decomposition $A \cong A_0 \times A_l \times A_r \times B$, where $A_l := X$, $A_r := Y$, $B := G$ and A_0 denotes the fiber of ρ at 0. But the direct proof of Proposition 4.1.1 is much simpler.

Returning to an arbitrary complete irreducible semigroup (S, μ) , the decomposition of $(A(S), A(\mu))$ obtained from that proposition is given by

$$A(S) = A_0 \times A(X) \times A(Y) \times G,$$

where A_0 denotes the fiber at 0 of $A(\rho) : A(S) \rightarrow A(X \times G \times Y) \cong A(X) \times A(Y) \times G$.

(ii) As a direct consequence of Theorem 4.3.1, every algebraic semigroup law on a complete irreducible curve is either trivial or the group law of an elliptic curve. This yields an alternative proof of part of the classification of irreducible algebraic semigroups of dimension 1 (Theorem 4.2.1); but in fact, both arguments make a similar use of the rigidity lemma.

4.4 Endomorphisms of complete varieties

In this subsection, we obtain the following rigidity result which is probably known, but for which we could not locate any reference.

Proposition 4.4.1. *Let X be a complete variety, T a connected variety, and $\varphi : X \times T \rightarrow X$ a morphism; for any $t \in T$, denote by φ_t the endomorphism of X such that $\varphi_t(x) = \varphi(x, t)$. If φ_{t_o} is an automorphism for some $t_o \in T$, then φ_t is an automorphism for any $t \in T$.*

Proof. We first show that the fibers of φ_t are finite for any $t \in T$. Assuming the contrary, we may find a complete irreducible curve $C \subseteq X$ such that φ sends $C \times \{t\}$ to a point for some $t \in T$. By the rigidity lemma, it follows that φ sends $C \times \{t\}$ to a point for all $t \in T$. Taking $t = t_o$, we get a contradiction.

Since each φ_t is proper, it follows that φ_t is finite and surjective. In fact, the morphism

$$\Phi := \varphi \times \text{id} : X \times T \longrightarrow X \times T, \quad (x, t) \longmapsto (\varphi(x, t), t)$$

is also finite and surjective: indeed, its fibers are clearly finite, and Φ is proper as the composition of the closed immersion $X \times T \rightarrow X \times X \times T$, $(x, t) \mapsto (x, \varphi(x, t), t)$ and of the projection $p_{23} : X \times X \times T \rightarrow X \times T$, $(x, y, t) \mapsto (y, t)$.

We now show that φ_t is an automorphism for all t in some neighborhood of t_o . This assertion is proved in [Ko95, Lem. I.1.10.1]; we recall the argument for completeness. Since Φ is proper, the sheaf $\Phi_*(\mathcal{O}_{X \times T})$ is coherent; it is also flat over T , since Φ lifts the identity of T . Moreover, the natural map $\Phi^\# : \mathcal{O}_{X \times T} \rightarrow \Phi_*(\mathcal{O}_{X \times T})$ induces an isomorphism $\Phi_{t_o}^\# = \varphi_{t_o}^\# : \mathcal{O}_X \rightarrow (\varphi_{t_o})_*(\mathcal{O}_X)$. In view of a version of Nakayama's lemma (see [Ko95, Prop. I.7.4.1]), it follows that $\Phi^\#$ is an isomorphism over a neighborhood of t_o . Hence the finite surjective endomorphism Φ is an automorphism over this neighborhood.

Thus, the set of points of T at which φ is an automorphism is open. To complete the proof, it suffices to show that this set is closed. For this, we may assume that T is an

irreducible curve; replacing T with its normalization, we may further assume that T is nonsingular. By shrinking T , we may finally assume that it has a point s such that φ_t is an automorphism for all $t \in T \setminus \{s\}$; we have to show that φ_s is an automorphism as well.

If X is normal, then so is $X \times T$; moreover, the above endomorphism Φ is finite and birational, and hence an automorphism. Thus, every φ_t is an automorphism.

For an arbitrary X , consider the normalization $\eta : \tilde{X} \rightarrow X$. Then Φ lifts to an endomorphism $\tilde{\Phi} : \tilde{X} \times T \rightarrow \tilde{X} \times T$, which is an automorphism by the above step. In particular, φ_s lifts to an automorphism $\tilde{\varphi}_s$ of \tilde{X} . We have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\varphi}_s} & \tilde{X} \\ \eta \downarrow & & \eta \downarrow \\ X & \xrightarrow{\varphi_s} & X \end{array}$$

and hence a commutative diagram of morphisms of sheaves

$$\begin{array}{ccc} \mathcal{O}_X & \longrightarrow & (\varphi_s)_*(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ \eta_*(\mathcal{O}_{\tilde{X}}) & \longrightarrow & \eta_*(\tilde{\varphi}_s)_*(\mathcal{O}_{\tilde{X}}). \end{array}$$

Moreover, the bottom horizontal map in the latter diagram is the identity (as $(\tilde{\varphi}_s)_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{\tilde{X}}$), and the other maps are injective. Thus, $\mathcal{O}_X \subseteq (\varphi_s)_*(\mathcal{O}_X) \subseteq \eta_*(\mathcal{O}_{\tilde{X}})$, and hence the iterates $(\varphi_s^n)_*(\mathcal{O}_X)$ form an increasing sequence of subsheaves of $\eta_*(\mathcal{O}_{\tilde{X}})$. As the latter sheaf is coherent, we get

$$(\varphi_s^n)_*(\mathcal{O}_X) = (\varphi_s^{n+1})_*(\mathcal{O}_X) \quad (n \gg 0).$$

Since φ_s is finite and surjective, it follows that $\mathcal{O}_X = (\varphi_s)_*(\mathcal{O}_X)$ and hence that φ_s is an isomorphism. \square

Corollary 4.4.2. *Let M be a complete algebraic monoid. Then $G(M)$ is a union of connected components of M .*

Proof. Let T be a connected component of M containing a unit t_0 . Applying Proposition 4.4.1 to the morphism $\mu : M \times T \rightarrow M$, we see that the map $x \mapsto xt$ is an isomorphism for any $t \in T$. Likewise, the map $x \mapsto tx$ is an isomorphism as well. Thus, t has a left and a right inverse in M , and hence is a unit. So T is contained in $G(M)$.

Alternatively, we may deduce the statement from Theorem 4.2.1: indeed, $G(M)$ contains no subgroup isomorphic to \mathbb{G}_a or \mathbb{G}_m , since the latter do not occur as unit groups of complete irreducible monoids. By Chevalley's structure theorem, it follows that the reduced neutral component of $G(M)$ is an abelian variety; thus, $G(M)$ is complete. But $G(M)$ is open in M , hence the assertion. \square

Corollary 4.4.3. *Let S be a complete connected semigroup, and $e \in E(S)$. Then eSe is an abelian variety. In particular, e is minimal.*

Proof. Clearly, eSe is a complete connected algebraic monoid; hence it coincides with its unit group by the above corollary. \square

Remarks 4.4.4. (i) The rigidity statement of Proposition 4.4.1 still holds with T being replaced by any connected scheme of finite type, since the proof adapts readily to that setting. We sketch a scheme-theoretic interpretation of that statement, when X is projective. Then there exists a quasiprojective scheme $\text{End}(X)$ which represents the endomorphism functor of X , i.e., for any scheme T , the set of T -points $\text{End}(X)(T)$ is the set of endomorphisms of $X \times T$ over T ; also, each connected component of $\text{End}(X)$ is of finite type. Moreover, there exists an open subscheme $\text{Aut}(X)$ of $\text{End}(X)$ such that $\text{Aut}(X)(T)$ is the set of automorphisms of $X \times T$ over T for any scheme T ; finally, $\text{End}(X)$ is an algebraic monoid scheme, and $\text{Aut}(X)$ is its unit group scheme (see e.g. [Ko95, §I.1] for these results). Now Proposition 4.4.1 means that $\text{Aut}(X)$ is open and closed in $\text{End}(X)$. In other words, $\text{End}(X)$ is a union of connected components of $\text{Aut}(X)$.

For an arbitrary complete variety X , the automorphism functor defined as above is represented by a group scheme $\text{Aut}(X)$; moreover, each connected component of $\text{Aut}(X)$ is of finite type (see [MO67, Thm 3.7] for these results). We do not know if $\text{End}(X)$ is representable in this generality; yet the above interpretation of Proposition 4.4.1 still makes sense in terms of functors.

(ii) Let X and T be complete varieties, where T is irreducible, and let $\mu : X \times T \rightarrow X$ be a morphism such that $\mu(x, t_o) = x$ for some $t_o \in T$ and all $x \in X$. Then by Proposition 4.4.1, the map $\mu_t : x \mapsto \mu(x, t)$ is an automorphism for any $t \in T$. This yields a morphism of schemes

$$\varphi : T \longrightarrow \text{Aut}(X), \quad t \longmapsto \mu_t$$

such that $\varphi(t_o)$ is the identity. Hence φ sends T to the neutral component $\text{Aut}^o(X)$. Consider the subgroup G of $\text{Aut}^o(X)$ generated by the image of T ; then G is closed and connected by [DG70, Prop. II.5.4.6], and hence is an abelian variety. In loose words, the morphism μ arises from an action of an abelian variety on X .

(iii) Let X be a complete irreducible variety, and $\mu : X \times X \rightarrow X$ a morphism such that $\mu(x, x_o) = \mu(x_o, x) = x$ for some $x_o \in X$ and all $x \in X$. Then the above morphism $\varphi : X \rightarrow \text{Aut}^o(X)$ satisfies $\varphi(x)(x_o) = x$, and hence is a closed immersion; we thus identify X to its image in $\text{Aut}^o(X)$. As seen above, X generates an abelian subvariety G of $\text{Aut}^o(X)$. The natural action of G on X is transitive, since the orbit Gx_o contains $Xx_o = X$. Thus, X itself is an abelian variety on which G acts by translations. Moreover, since $Gx_o = Xx_o$, we have $G = XG_{x_o}$, where G_{x_o} denotes the isotropy subgroup scheme of x_o . As G is commutative and acts faithfully and transitively on X , this isotropy subgroup scheme is trivial, i.e., $G = X$. In conclusion, X is an abelian variety with group law μ and neutral element x_o . This result is due to Mumford, see [Mu74, Chap. II, §4, Appendix].

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